STOCHASTIC REPUTATION CYCLES

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Abstract. This paper studies a model of reputation-building in which the reputation of a firm is treated as capital stock that accumulates by past investments, depreciates when there is no investment, and has a persistent effect on future payoffs. The setting is a discrete-time discounted stochastic game between a long-run firm and a sequence of short-run buyers where the firm's reputation is the state variable. Under a class of transition rules, if actions are taken frequently enough, there is a unique stationary Markov equilibrium, which is characterized by a reputation-building stage, a reputation-exploitation stage and a possible reputation-absorbing stage. For low levels of reputation, the firm randomizes between investing and not investing, and the buyers randomize between buying and not buying. The firm always has incentive to build reputation even if the stock reaches the lowest level. For high levels of reputation, the buyers buy with probability one and the firm exploits the reputation by not investing. Reputation moves cyclically between these two stages, so reputation is a long-run phenomenon. Under certain circumstances, there is an extra stage, a reputation-absorbing stage. If the firm's reputation is very low, the firm loses the incentive to invest, thus reputation eventually declines to the lowest level which is an absorbing state.

1. Introduction

Since the seminal work of Kreps and Wilson (1982) and Milgrom and Roberts (1982), it has been well understood that reputation considerations are important in long-term relationships. In this literature, reputation is captured by the belief of the uninformed party as to the type of the informed party. Specifically, it is typically assumed that the informed party is of two types, a "commitment" type and a "normal" type, where the commitment type is not

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strategic and follows a pre-specified rule of behavior, with the behavior of the normal type being the object of the analysis. The uninformed party does not know the type of the other party and updates beliefs using past histories. We then say that the informed party has a "reputation" (of being the commitment type) if the probability assigned to the commitment type is not zero.

When actions can be observed, as soon as an opportunistic action that would never be taken by the commitment type is observed, the reputation of being a commitment type vanishes to zero by Bayesian updating and has no chance of bouncing back. When actions cannot be perfectly observed, even though opportunistic behavior does not totally ruin reputation, the type is eventually learned, so reputation is again a short-term phenomenon (Cripps, Mailath and Samuelson, 2007). However, in reality, reputation might be sustainable in the long run, as illustrated by many successful reputation recovery stories.[1] The evidence from those stories is that reputation only gets tarnished rather than vanishing. Moreover, its negative effect is felt through sales instead of prices, and reputation can be eventually restored. For instance, in 2010, the safety recalls for brake and accelerator problems tarnished Toyota's high reputation ranking, and the reputation damage caused sale reduction (Shin, Richardson, and Soluade, 2012). However, after three years of a gradual reputation-recovery process, Toyota has bounced back to become one of the most highly regarded companies in the U.S. by 2013.[2]

There are papers in the reputation literature that obtain reputation as a long-run phenomenon, but they resort to exogenous uncertainty such as replacement of types (Holmstrom, 1999; Mailath and Samuelson, 2001; Phelan, 2006; Ekmekci, Gossner and Wilson, 2012), limited record of history (Liu and Skrzypacz, 2009; Liu, 2011; Monte, 2013) or information censoring (Ekmekci, 2011). However, in many situations, reputation may be restored by a firm's ability to endogenously improve it, instead of the exogenous reasons mentioned above.

[1]Dietz and Gillespie (2012) present six case studies about The BBC, Siemens, Mattel, BAE Systems, Severn Trent and Toyota. Sharon Beder (2002) studies Nike's successful reputation recovery from criticism over its poor labour and environmental standards. See also the good reaction to social media crisis by Kitchenaid, DKNY and Burger King (http://oursocialtimes.com/6-examples-of-social-media-crises-what-can-we-learn/).

[2] 2008 - 2014 Harris Poll Reputation Quotient (RQ), from Harris Interactive. The ranking of Toyota among the most visible companies in the U.S. from 2008 to 2014 is 10th, 20th, 20th, 43rd, 31st, 19th and 21st.

For example, Toyota's recovery from the lost reputation was due to its program of thorough reforms such as new safety and quality control systems. Similarly, Siemens overhauled its structures, leadership, processes and culture after the accusation of its systematic bribery in 2006.

Recent papers capture the idea that a firm's reputation is accumulated by past efforts. In Board and Meyer-ter-Vehn (2013) and Dilme (2012), reputation is treated as a belief of product quality that can be changed by a firm's past investments. Faced with new information, reputation goes through discontinuous jumps, relative to continuous drifts when there is no new information. Furthermore, reputation only brings a premium to the price since the price is the expected quality of the product and the buyers have unit demand for the product. However, the reputation stories mentioned before suggest that reputation depreciates instead of jumping discontinuously, and its impact is mainly felt through sales instead of prices. In Halac and Prat (2014), firm's reputation of monitoring the workers' performance is modeled as a belief of the quality of managerial practice, an intangible and imperfectly observable asset. Upon the arrival of a perfect good news about the firm's quality, reputation displays discontinuous jumps. Huang and Li (2014) modeled mutual fund's reputation as the market's belief whether the fund has profitable information that can be costly acquired but may depreciate. Unlike Board and Meyer-ter-Vehn (2013), manager's reputation either goes up or down smoothly in each period. Bohren (2011) studies a class of stochastic games in which the actions of a long-run player have a persistent effect on future payoffs. Past effort is considered as the source of reputation, which influences the short-run buyers' willingness to buy. However, Bohren (2011) has little power for explaining long-run reputation effects. There is a key assumption that there are absorbing states in the boundary, so reputation can be permanent only when it reaches the boundary with zero probability, but it is not clear when this holds.[3] If reputation starts from the boundary points, the long-run player loses the incentives to build reputation, thus reputation is a short-run phenomenon.

Following the idea of action persistence in Bohren (2011), this paper models reputation as capital stock that is smoothly accumulated by investment and depreciates when there is no investment. The following are four examples in which reputation behaves like capital stock.

[3]For the unbounded state space, the assumption says that the state variable cannot pull the state back to a region with non-negligible incentives fast enough when the state variable becomes very large or very small.

(1) High quality can be treated as reputation of firms. For example, Toyota enjoyed a high reputation because it had made continuous R&D to guarantee reliable vehicles. The recalls in 2010 were due to a design flaw, which had nothing to do with its manufacturing stock.[4] It seems plausible to consider this stock as the main determinant of Toyota's reputation, so reputation would only suffer a decline instead of a total ruin after the design flaw.
(2) Goodwill is an intangible asset which represents the extra value ascribed to a company by virtue of its brand and reputation.[5] Goodwill is represented by the value of a company's brand name, solid customer base, good employee relations and any patents or proprietary technology, which produce income in the future. In order to acquire a high goodwill, a company needs to make consistent investments such as advertising, developing the workforce, and increasing the customer base.
(3) Human capital stock is treated as a worker's reputation (Camargo and Pastorino, 2001). A worker receives costly on-the-job training and learning-by-doing to accumulate human capital, which influences his or her future productivity and changes the experience in the future labor market.
(4) Knowledge stock can be considered as the reputation of the economy in the endogenous growth models (Romer, 1986, 1989; Jones and Manuelli, 1990; King and Rebelo, 1990). Higher accumulation of knowledge will enhance the future productivity of the economy. Governments avoid short-sighted high taxation because it hurts the production of knowledge and hence long-run economic growth.

By modeling reputation as capital stock that is endogenously influenced by past investments, this paper delivers reputation cycles that persist in the long run, and is characterized by phases of reputation building and exploitation. This is in contrast with the temporary reputation effects observed in traditional belief-based models as well as in Bohren (2011). Furthermore, prices are implicitly fixed, so reputation has an impact only on sales, which fits several reputation stories mentioned before.

[4]http://news-releases.uiowa.edu/2010/february/020510toyota-researcher.html.

[5]The goodwill, the bad and the ugly, *Economist*, Jan 22nd 2009. Indeed, goodwill is added to the combined entity's balance sheet during mergers and acquisitions.

This paper relates to the large literature on existence of stationary equilibria in discrete-time discounted stochastic games which started with Shapley (1953) and is still active (See Levy and McLennan (2015) for further reference). This paper studies a specific discrete-time discounted stochastic game and characterizes the unique equilibrium among all stationary Markov equilibria if actions are taken frequently enough, given any fixed discount rate. Focusing on perfect public equilibria (PPE) other than stationary equilibria, several folk theorems characterize the equilibrium payoffs for discrete-time discounted stochastic games as players are patient enough.[6] Pęski and Wiseman (2014, 2015) study discrete-time discounted dynamic stochastic games where transition rules depend on the length of the period, and characterize the PPE payoffs if the length of the period shrinks, given fixed discount rate. Related to Pęski and Wiseman (2014), this paper studies transition rules that depend on the length of the period: the magnitude of the state transition is proportional to the length of the period.

Formally, we consider a discrete-time discounted stochastic game between a long-run player (henceforth firm) and a sequence of short-run players (henceforth buyers). In each period, a buyer decides whether to buy the firm's product or not, and the firm can either invest in its reputation or not. Period payoffs depend on the current actions of the players and on the reputation stock, which is the state variable. The reputation stock evolves according to a transition rule that depends on the firm's decisions. The firm discounts the future with a constant discount factor. Restricting to stationary Markov equilibrium, we study the dynamics of reputation under different transition rules. In particular, we determine when reputation is not short-lived.

We consider transition rules that are "local" in order to capture the spirit that reputation accumulates and depreciates smoothly as capital stock, instead of presenting discontinuous jumps as a belief. Heuristically, we do not allow for drastic jumps in the stock as a result of investment or lack of it, which in this model means that the next period's stock is at most one unit apart from the current stock. Two types of rules can illustrate the qualitative properties of all "local" transition rules. The first type is called *one-step transition rules*, in which only the firm has the power of controlling the reputation in the following way: investing leads to a one-step increase while not investing causes a one-step depreciation of reputation, with

[6] See Dutta (1995), Fudenberg and Yamamoto (2011), and Hörner, Sugaya, Takahashi, and Vieille (2011).

the possibilities that investing may cause one-step depreciation and not investing results in one-step increase with small probability.

The second type is called an *augmented one-step transition rule*, in which both the firm and the buyers can influence the reputation. If the buyers buy, this rule is the same as the *one-step transition rule* without noises. However, if the buyers choose not to buy, the firm has no chance of building reputation and the reputation remains the same. That the firm should control the stock is obvious, because it is the result of investments by the firm. However, we also allow the buyers to influence it because of practical considerations. A firm's word-of-mouth advertisement today may not effectively improve its reputation if the buyers do not buy, experience the good and give high customer ratings to influence the decision of future buyers. Workers have no chance of learning-by-doing without being hired in the first place, likewise an economy cannot invest in knowledge if the public does not produce any consumption good. Furthermore, in order to qualitatively investigate the role that the buyers play in determining the reputation, it is enough to investigate the *augmented one step transition rule* in which the buyers are given the maximal power of influencing the reputation because they can take away all incentives for the firm to build reputation if they choose not to buy. Once we figure out how this maximal power changes the equilibrium behavior, equilibrium behavior under other transition rules that give intermediate power to the buyers will yield intermediate results.

Finally, to facilitate a comparison with the belief-based reputation literature, we also study *lower-bound transition rules*, in which it is possible for reputation stock to jump to the lower-bound if the firm does not invest. Among all rules that the buyers have no power of controlling the reputation, *one step transition rules* and *lower-bound transition rules* are two extreme cases with respect to the downward transition. Therefore, equilibrium behavior under these two transition rules sheds light on the equilibrium behavior in any other transition rule with intermediate downward transition.

Under any of the transition rules described above, when actions are taken frequently enough, there is a unique stationary Markov equilibrium, which is characterized by a reputation building stage, a reputation-exploitation stage and a possible reputation-absorbing stage. When reputation stock is lower than a threshold, there is a reputation-building stage in which players randomize, and the firm always has incentive to invest even if the stock reaches the lowest level. For high levels of reputation which are at or above a threshold[7], the buyers buy for sure and the firm exploits its reputation by not investing. Therefore, the reputation stock goes up and down (as players randomize) as long as it is below the threshold. Once it is above the threshold, it goes back down as the firm does not invest. This process repeats ad infinitum. However, if the threshold is too high, then there is an extra stage, a reputation-absorbing stage: for very low levels of reputation, the firm loses the incentives to invest, thus reputation eventually declines to the lowest level which is an absorbing state. In all, if the threshold is low enough, reputation keeps moving back and forth and never stays at a certain state. Therefore, reputation is a long-run phenomenon, which fits the successful reputation recovery stories including Toyota's story of recall and its subsequent come back. If the threshold is high enough, the firm never has the incentive to build reputation and reputation will be finally stagnant at the lowest level.

Comparison with Bohren (2011). It is important to compare this paper to Bohren (2011), because both papers model reputation in a stochastic game framework. We analyze a discrete-time model with a product-choice stage game and characterize a unique stationary Markov equilibrium when actions are taken frequently enough. There are some key differences between the two papers. (i) Permanent reputation. In Bohren (2011), there is a key assumption that there are absorbing states at the boundary of the state space to guarantee the uniqueness of the Markov equilibrium, and reputation can be permanent only when the state reaches the boundary with zero probability, but it is not clear when this holds. When the state starts from the boundary, the long-run player loses the incentive to invest and reputation is stagnant. In our model, we do not need this assumption and explicitly characterize the necessary and sufficient condition for the existence of an absorbing state. If the firm's investment cost is low enough and the firm is patient enough, there is no absorbing state and reputation is a permanent phenomenon. (ii) Brownian signal. In order to guarantee the existence of a Markov equilibrium, Bohren (2011) requires the imperfect signal to be Brownian and the volatility of the state variable to be bounded away from zero at interior points. Having infinite variation on any small time interval, Brownian motion might be suitable to describe the path of prices in the market. However, reputation, as a capital shock in this

[7] The threshold is determined by firm's discount factor, firm's investment cost and transition rules. See Corollary 3.1.

paper, is less likely to display infinite variation as in a Brownian motion. Therefore, the assumption of smooth transition rules in the sense that reputation moves at most one-step up or down seems more appropriate. (iii) Bohren (2011) can identify the condition to guarantee that the only Perfect Public Equilibrium (PPE) is Markovian by combing frequent actions with noisy Brownian information. No such result is available in our model. We show the uniqueness of equilibrium among all Markov equilibria and there may be non-Markovian equilibria.[8] In all, this paper provides a rationale for permanent reputation in a stochastic game setting that uses the typical product-choice stage game and hence is easily comparable to he belief-based reputation literature. Moreover, unlike Bohren (2011) and consistent with the idea that reputation is built smoothly as capital stock. However, this comes at a cost because we only focus on the uniqueness of stationary Markov equilibrium, and Bohren (2011) has the uniqueness of all PPE.

2. Model

We study a discrete-time discounted stochastic game where a long-run player (henceforth the firm) plays against an infinite sequence of short-run players (henceforth the buyers). Time is discrete and indexed by $t = 0, \Delta, 2\Delta \ldots, \infty$. Δ is the length of each period. In later sections, we will analyze the case where Δ is small and also the limit as $\Delta \to 0$. A buyer who arrives at time t plays a stage-game with the firm, then exits and does not come back. The firm discounts future payoffs by $\delta = e^{-r\Delta}$ and maximizes the expected sum of discounted payoffs. The buyers only care about their stage-game payoffs.

Reputation of the firm is modeled as a state variable X, which affects only the stage-game payoffs of the buyers. The state space $\mathscr{X}_\Delta$ is $\{0, \Delta, 2\Delta, \ldots\}$, which means that the shift of reputation X is proportional to the time interval Δ. This captures the idea that reputation building (or milking) is a smooth process if we restrict the maximal steps of reputation shift to be bounded in each period.

The stage game is a modified version of product-choice game in which the buyers' stage-game payoff depends on firm's reputation. In each period, the firm and a buyer move simultaneously. There are two pure actions for the firm: I and NI, which represent investing

[8]In our model, if the volatility of the state variable is bounded away from zero as actions are taken frequently enough, it is not clear whether there is non-Markovian equilibrium or not.

and not investing. There are two pure actions for the buyer: B and NB, which represent buying and not buying.

The following is an example of a stage-game payoff matrix that illustrates a product-choice game that we will study. The row player is the firm and the column player is the buyer.

	B	NB
I	$1, \lambda + (1-\lambda)X$	$-\frac{1}{2}$, 0
NI	$2, -\lambda + (1-\lambda)X$	0,0

Notice the following properties of the firm's stage-game payoff. (i) The firm's stage-game payoff is not directly influenced by reputation X.[9] (ii) The firm is better off if the buyer buys. (iii) It is a dominant strategy for the firm not to invest. The firm's investment cost is 1 if the buyer buys and $\frac{1}{2}$ if the buyer does not buy. Therefore, the expected investment cost is increasing in the buyer's probability of buying, which is called *submodularity.* (iv) The firm prefers (I, B) to (NI, NB), which means that the firms is better off if committing to investing is possible. Next, we describe four properties of the buyer's stage-game payoff. (i) The buyer's stage-game payoff is increasing in X if the buyer buys, which means that reputation is valuable for the buyer. (ii) The buyer is better off if the firm invests. (iii) The buyer prefers to buy if the firm invests, and gets the same payoff 0 if the firm does not invest. (iv) If $X \geq X^* = \frac{\lambda}{1-\lambda}$, it is a weakly dominant strategy for the buyers to buy. If $X < X^*$, then there is a probability of investing for the firm that makes the buyer indifferent between B and NB, which is denoted as $a^*(X) = \frac{1}{2} - \frac{1-\lambda}{2\lambda}X$.

Assumption 2.1-2.5 make the above statements formal. The firm's stage-game payoff is $g_1 : \{I,\ NI\} \times \{B, NB\} \mapsto \mathbb{R}$. The buyers' stage-game payoffs also depend on the state variable $X \in \mathbb{R}^+$: $g_2 : \{I,\ NI\} \times \{B, NB\} \times \mathbb{R}^+ \mapsto \mathbb{R}$.

Assumption 2.1: $g_1(NI, B) \geq g_1(I, B)$, $g_1(NI, NB) \geq g_1(I, NB)$; $g_1(I, B) > g_1(I, NB)$, $g_1(NI, B) > g_1(NI, NB)$; $g_1(I, B) > g_1(NI, NB)$.

Assumption 2.2: $g_1(NI, B) - g_1(I, B) > g_1(NI, NB) - g_1(I, NB)$.

[9]Justified by several real-world reputation stories as we saw in the introduction, this paper models the impact of reputation on sales instead of prices, which may directly influence the firm's payoff. A model that captures sales and prices is left for future research.

Assumptions 2.1-2.2 describe the stage-game payoff of the firm. Assumption 2.1 tells us that in a stage-game, the firm prefers not to invest and the firm is better off if the buyer buys. Moreover, the firm prefers cooperation (I, B) to noncooperation (NI, NB), which means that the firm has an incentive to build reputation. Assumption 2.2 is the *submodularity* of the firm's payoff, which characterizes the conflict between the firm and the buyer.

Assumption 2.3: $g_2(I, B, X) > g_2(NI, B, X)$, $g_2(I, B, X) > g_2(I, NB, X)$ for any X. $g_2(I, NB) = g_2(NI, NB)$.

Assumption 2.4: $g_2(I, B, X)$ and $g_2(NI, B, X)$ are strictly increasing in X.

Assumption 2.5: There is $X^* > 0$ such that $X \geq X^*$ imples $g_2(NI, B, X) \geq g_2(NI, NB)$ and $X < X^*$ implies $g_2(NI, B, X) < g_2(NI, NB)$.

Assumptions 2.3-2.5 describe the stage-game payoff of the buyer. Assumption 2.3 tells us that the buyer prefers to buy if the firm invests, and gets the same payoff if the firm does not invest. Moreover, the buyer wants the firm to invest. Assumption 2.4 means that reputation is valuable for the buyer, because higher reputation yields higher payoff for the buyer if the buyer buys. Assumption 2.5 tells us that if $X \geq X^*$, it is a weakly dominant strategy for the buyer to buy, which means that the buyer prefer to buy independently of the firm's current behavior if the firm has accumulated enough reputation in the past. If $X < X^*$, then there is a probability of investing for the firm that makes the buyers indifferent between B and NB:

$$a^*(X) \equiv \frac{g_2(NI, NB) - g_2(NI, B, X)}{g_2(I, B, X) - g_2(NI, B, X)}.$$

Observe that $a^*(X)$ is decreasing in X.

Let $a \in [0, 1]$ denote the mixed strategy of the firm: the probability of playing I. Let $y \in [0, 1]$ denote the mixed strategy of the buyer: the probability of playing B. For a given pair of mixed actions (a, y), let $g_1(a, y)$ and $g_2(a, y, X)$ denote the expected stage payoffs of the firm and the buyers.

Assume that the firm and the buyers can observe all the past history: the past actions of the long-run firm and the short-run buyers, and the state variable X.

Finally, we specify the transition rules of state variable X, which characterize how the current actions have a persistent impact on the future payoffs of the buyers. We consider

Markov transition rules represented by a transition probability

$$P : \{I, NI\} \times \{B, NB\} \times \mathscr{X}_\Delta \mapsto \Delta(\mathscr{X}_\Delta).$$

Given the firm's action $f \in \{I, NI\}$, the buyer's action $b \in \{B, NB\}$ and the current state variable X, $P(f, b, X)$ is the probability of the state X' in the next period. Given mixed strategy (a, y) and the current state X, the probability of next state X' is

$$P(a, y, X) = ayP(I, B, X) + a(1-y)P(I, NB, X) + (1-a)yP(NI, B, X) + (1-a)(1-y)P(NI, NB, X).$$

3. Equilibrium Analysis

We consider *stationary Markov Equilibria* in which both the firm and the buyers play *stationary Markov strategies*. Denote $(a(X), y(X))$ as the mixed actions of the firm and the buyers which only depend on the current state X. Define $V(X)$[10] as the firm's continuation value at state X.

Definition 3.1: $(a(X), y(X), V(X))$ is a *stationary Markov Equilibrium* if for all X,

$$\begin{aligned} V(X) &= \max_{a \in [0,1]} (1-\delta) g_1(a, y(X)) + \delta E_P V(X'). \\ a(X) &\in \operatorname*{argmax}_{a \in [0,1]} (1-\delta) g_1(a, y(X)) + \delta E_P V(X'). \\ y(X) &\in \operatorname*{argmax}_{y \in [0,1]} g_2(a(X), y, X), \end{aligned}$$

$$s.t.\ P = P(a, y, X).$$

We are interested in two kinds of stationary Markov equilibria: *non-absorbing equilibria* and *quasi-absorbing equilibria*. In a *non-absorbing equilibrium*, the buyers buy at state 0: $y(0) > 0$, the firm always has incentive to invest at state 0: $a(0) > 0$ and there is no absorbing state. Moreover, there are two reputation stages as follows:

Define K as the smallest integer satisfying $K\Delta > X^*$, that is $K = \lfloor \frac{X^*}{\Delta} \rfloor + 1$.

(1) Reputation-building stage: $0 \le k \le K-1$. The firm invests with positive probability: $a(k\Delta) \ge a^*(k\Delta)$, and the buyers buy with positive probability $y(k\Delta) > 0$.
(2) Reputation-exploitation stage: $k \ge K$. The firm does not invest and the buyers buy, i.e. $y(k\Delta) = 1$ and $a(k\Delta) = 0$.

[10] $V(X)$ is bounded above by $g_1(NI, B)$, which is the highest stage payoff that the firm can get, so the transversality condition is satisfied.

In the *quasi-absorbing equilibrium*, the buyers do not buy at state 0: $y(0) = 0$, the firm does not invest at state 0: $a(0) = 0$.[11] There are three reputation stages. There exists an integer $\bar{K} > 0$ such that

(1) Reputation-absorbing stage: $0 \leq k \leq K - \bar{K} - 1$. The firm does not invest and the buyers do not buy: $a(k\Delta) = y(k\Delta) = 0$.
(2) Reputation-building stage: $K - \bar{K} \leq k \leq K - 1$. The firm invests with positive probability: $a(k\Delta) \geq a^*(k\Delta)$, and the buyers buy with positive probability $y(k\Delta) > 0$.
(3) Reputation-exploitation stage: $k \geq K$. The firm does not invest and the buyers buy, i.e. $y(k\Delta) = 1$ and $a(k\Delta) = 0$.

3.1. **One-step transition rules.** In this section, we focus on a specific class of transition rules: *one-step transition rules*, which capture the ideas that reputation accumulates and depreciates smoothly, as the maximal step of reputation shift is Δ. Moreover, only the firm has the power of controlling reputation transitions.

(1) If the firm invests, then the probability that $X' = X + \Delta$ is $1 - q$ and the probability that $X' = \max\{X - \Delta, 0\}$ is q:

$$P(I, X) = \begin{cases} 1 - q & X' = X + \Delta \\ q & X' = \max\{X - \Delta, 0\}. \end{cases}$$

(2) If the firm does not invest, then the probability that $X' = \max(X - \Delta, 0)$ is $1 - p$ and the probability that $X' = X + \Delta$ is p:

$$P(NI, X) = \begin{cases} 1 - p & X' = \max\{X - \Delta, 0\} \\ p & X' = X + \Delta. \end{cases}$$

(3) $P(a, X) = aP(I, X) + (1 - a)P(NI, X)$.

Given that the firm does not invest, p is the probability of reputation up-shift. Given that the firm invests, q is the probability of reputation down-shift. If $p = q = 0$, the *one-step transition rule* is deterministic: investing leads to high reputation and not investing leads to lower reputation. In the deterministic case, the marginal benefit of investing reaches the

[11] 0 is an absorbing state if and only if not investing cannot lead to an increase of reputation. For example, in the *one-step transition rules* described below, 0 is an absorbing state if and only if $p = 0$. If p is small enough, the probability of escaping from state 0 is small. That is the reason why we call this kind of equilibrium *quasi-absorbing equilibrium.*

highest level. Higher p and q, considered as a measure of shocks, implies lower benefit of investing.

Define three payoff parameters A, A_{pq} and γ as below:

$$A=\frac{g_1(1,1)-g_1(1,0)}{g_1(0,1)-g_1(0,0)},\ A_{pq}=\frac{(1-p)A-q}{1-q-Ap},\ \gamma=\frac{g_1(0,0)-g_1(1,0)}{g_1(0,1)-g_1(0,0)}.$$

The parameter A captures the *submodularity* of the firm's payoffs. Higher A means a low degree of submodularity, thus a higher intensity of conflict between the firm and the buyers. A_{pq} measures both the shocks p and q as well as submodularity. The parameter γ captures the investment cost if the buyer does not buy. By Assumption 2.1, $\gamma < A$. By Assumption 3.1, $A_{pq} \in (0,1)$.

Assumption 3.1: $p+q<1$, $\frac{q}{1-p} < A < \frac{1-q+q^2}{1-pq}$.

Assumption 3.1 holds for small p and q. $A > \frac{q}{1-p}$ guarantees that the buyers can provide enough incentives for the firm to invest. $p+q<1$ implies that investing will leads to high probability of reputation up-shift than not investing. $A < \frac{1-q+q^2}{1-pq}$ is a technical condition to guarantee the uniqueness of the stationary Markov equilibrium.

Theorem 3.1 characterizes the equilibrium behavior under *one-step transition rules* if actions are taken frequently enough.

Theorem 3.1. *Under Assumptions 2.1-2.5, 3.1 and one-step transition rules, for each* $p \geq 0, q \geq 0$, *there exists a* $\bar{\Delta}_{pq} > 0$ *such that for all* $\Delta < \bar{\Delta}_{pq}$, *any stationary Markov equilibrium is characterized as follows. There exist* $\bar{K}_{pq} > 0$ *and* $M_{pq} > 0$ *s.t.*

(1) If $K \leq \bar{K}_{pq}$, *the stationary Markov equilibrium is a non-absorbing equilibrium.*

(2) If $K \geq \bar{K}_{pq}+1$, *the stationary Markov equilibrium is a quasi-absorbing equilibrium.*

(3) If $\max(K-\bar{K}_{pq},0)+M_{pq} \leq k \leq K-1$, *the firm plays mixed strategy* $a(k\Delta)=a^*(k\Delta)$. *The buyers play mixed strategy* $y(k\Delta) \in (0,1)$ *given by* $y(k\Delta) \equiv z_k - \frac{\gamma}{1-A}$, *where* $z_{k+2} = \frac{1}{\delta}(1-A_{pq})z_{k+1} + A_{pq}z_k$.

Moreover, if $p=q=0$, *then the stationary Markov equilibrium is unique and* $M_{00}=0$.[12]

Theorem 3.1 states that the stationary Markov equilibrium can only be one of the two kinds of equilibria: *non-absorbing equilibria* and *quasi-absorbing equilibria.*

[12] If $p=q=0$ and $K \leq \bar{K}_{pq}$, then for state $k=1$, there are two possibilities. Define $\epsilon = \frac{1}{2\delta}(1-A+\sqrt{(1-A)^2+4A\delta^2})$. If $\epsilon^K(1+\epsilon)+(-\frac{A}{\epsilon})^K(1-\frac{A}{\epsilon}) > (\epsilon+\frac{A}{\epsilon})(1+\epsilon-\frac{A}{\epsilon})$, then $a(k\Delta)=a^*(k\Delta)$ and $y(k\Delta)\in(0,1)$. Otherwise, $a(k\Delta)=y(k\Delta)=1$.

If it is easy to reach the state in which the buyers buy for sure ($K \leq \bar{K}_{pq}$), then reputation cycle is characterized by a reputation-building stage and a reputation-exploitation stage. In the latter stage when reputation is high enough, the buyers' dominant strategies are to buy. Therefore, the buyers cannot reward the firm by increasing the probability of buying any more, thus there is no incentive for the firm to build reputation any more. In the former stage when the reputation is low, the buyers randomize between buying or not buying in order to make the firm indifferent between investing and not investing. The firm also needs to randomize in such a way ($a(k\Delta) = a^*(k\Delta)$) that so that the buyers are indifferent between buying or not buying. The firm never loses the incentive to invest because reputation can be exploited in the near future. Even if the reputation hits the lower bound 0, the firm still invests with positive probability so that reputation will never be trapped at the lower bound 0.

On the other hand, if it is difficult to reach the state in which the buyers buy for sure ($K \geq \bar{K}_{pq} + 1$), then there is one extra stage: a reputation-absorbing stage. For low states, the firm loses the incentive to invest because the long-term benefit of building a reputation is dominated by the short-term cost of investing. As a result, the buyers' best choice are not to buy. Reputation moves down stochastically to state 0. For intermediate states, the firm builds reputation with positive probability and reputation can move upward or downward. After numerous steps of upward shifts, reputation gets to the reputation-exploitation stage, in which the firm exploits the reputation since there is no need to build more reputation, and go downward back to the reputation-building stage. If $p = 0$, then after a long sequence of downward drifts, reputation reaches the reputation-absorbing stage and thus continue to go down all the way to the absorbing state 0. In all, the reputation stock will eventually reach the absorbing state 0 and stay there forever, thus reputation is only a short-run phenomenon. If $p > 0$, then not investing may lead to a one-step increases of reputation with probability p, thus there is a chance that reputation comes back from the reputation-absorbing stage to the reputation-building stage.

Given $p = q = 0$, there is a unique stationary Markov equilibrium, which is completely characterized by Theorem 3.1. If $p = q = 0$ does not hold, then we consider the first state of reputation-building stage: $X = \max(K - \bar{K}_{pq}, 0)\Delta$. If the state is away from the first state: $X \geq (\max(K - \bar{K}_{pq}, 0) + M_{pq})\Delta$, then there is a characterization of the equilibrium: both the buyers and the firm play strictly mixed strategies. However, there is no characterization of the equilibrium behavior around the first state: $\max(K - \bar{K}_{pq}, 0)\Delta \leq X \leq (\max(K - \bar{K}_{pq}, 0) + M_{pq})\Delta$, thus the uniqueness of the stationary Markov equilibrium is not guaranteed. We deal with this issue in Theorem 3.2 below, which shows that $M_{pq}\Delta \to 0$ as $\Delta \to 0$, thus the limiting equilibrium is unique at $X = \max(K - \bar{K}_{pq}, 0)\Delta$.

3.2. **The Limiting Equilibrium: $\Delta \to 0$.** It is useful to consider the limiting equilibrium,[13] when $\Delta \to 0$ because we can present an analytic solution with clearer expressions than the non-limiting result. Therefore, we can do a thorough analysis of the equilibrium behavior, as well as comparative statics in order to check how the equilibrium can be impacted by the parameters. Furthermore, we can analyze the condition which determines the existence of an absorbing state.

Theorem 3.2 describes the limiting equilibrium behavior as $\Delta \to 0$. Define $(a(X), y(X), V(X)) \equiv \lim_{\Delta \to 0, k\Delta \to X}(a(k\Delta), y(k\Delta), V(k\Delta))$. Define $y^*(X) \equiv e^{-r\frac{1-A}{1-2q+(1-2p)A}(X^*-X)}(1 + \frac{\gamma}{1-A}) - \frac{\gamma}{1-A}$, an exponentially increasing function which will be shown as the buyers' equilibrium behavior. Define the limit of the threshold $\bar{K}_{pq}\Delta$ as

$$\bar{X}_{pq} \equiv \frac{1}{r}\log(\frac{1-2q+(1-2p)A}{2(1-p-q)}\frac{1-A+\gamma}{\gamma})\frac{1-2q+(1-2p)A}{1-A}.$$

Theorem 3.2. *Under Assumptions 2.1-2.5, 3.1 and one-step transition rules, the unique stationary Markov equilibrium in the limit as $\Delta \to 0$ exists and the equilibrium strategy is described as follows:*

(1) If $X^* \leq \bar{X}_{pq}$, then the equilibrium is a non-absorbing equilibrium:

$$(a(X), y(X)) = \begin{cases} (a^*(X), y^*(X)) & 0 < X < X^* \\ (0, 1) & X \geq X^*. \end{cases}$$

(2) If $X^* > \bar{X}_{pq}$, then the equilibrium is a quasi-absorbing equilibrium:

$$(a(X), y(X)) = \begin{cases} (0, 0) & 0 < X < X^* - \bar{X}_{pq} \\ (a^*(X), y^*(X)) & X^* - \bar{X}_{pq} < X < X^* \\ (0, 1) & X \geq X^*. \end{cases}$$

[13]The limiting equilibrium is the limit of any sequence of equilibria with $\Delta > 0$, as $\Delta \to 0$.

Consider the equilibrium behavior in the reputation-building stage ($a(X) = a^*(X)$). As reputation X increases, the buyers raise the probability of buying $y(X)$ to make the firm randomize between investing and not investing; $y'(X)$, the growth rate of $y(X)$, also increases since the buyers need to increase the firm's benefit of investing to match with the larger investment cost (due to *submodularity*); the probability of investing $a(X) = a^*(X)$ declines since it is easier for the firm to make the buyers indifferent between buying and not buying.

In the *quasi-absorbing equilibrium* ($X^* > \bar{X}_{pq}$), $y(X)$ is not continuous at the $X^* - \bar{X}_{pq}$, the threshold between the reputation-absorbing stage and the reputation-building stage. Actually, $y(X^* - \bar{X}_{pq})$ does not exist since the variation of $y(k\Delta)$ does not vanish as $k\Delta \to X^* - \bar{X}_{pq}$ and $\Delta \to 0$. As a sequence, the value function $V(X)$ is not "smoothly pasted" at $X^* - \bar{X}_{pq}$, as $y(X)$ is not continuous at $X^* - \bar{X}_{pq}$. The value function $V(X)$ is "smoothly pasted" at X^*, as $y(X)$ is continuous at X^*.[14]

Corollary 3.3. *Under Assumptions 2.1-2.5, 3.1 and one-step transition rules,*
(1) $\bar{X}_{pq}$ is decreasing in (r, A^{-1}, γ, p, q).
(2) $y(X)$ is non-increasing in (r, A^{-1}, γ, p, q) for any X.
(3) $V(X)$ is non-increasing in (r, A^{-1}, γ, p, q) for any X.

Corollary 3.3 presents comparative-statics analysis in order to derive some testable implications from the model. First, we study the impact of payoff parameters A, b and γ on the equilibrium behavior. In the reputation-building stage, the buyers are less likely to buy the product (smaller $y(X)$), if the firm cares less about future (larger r), the conflict between the firm and the buyers becomes more serious (larger A^{-1}) and the investment cost increases (larger γ). All the above changes of parameters weaken the incentives for the firm to invest. In order to compensate the weakening of incentives, the buyers have to provide more incentive by raising the growth rate of $y(X)$. Because $y(X)$ reaches 1 at a given threshold X^*, a high growth rate in $y(X)$ implies a lower $y(X)$ at each given X. Consequently, larger (r, A^{-1}, γ) imply lower continuation value $V(X)$ for each X since the buyer are less likely to buy (lower $y(X)$). Larger (r, A^{-1}, γ) also imply smaller $\bar{X}_{pq}$, which means that it is more likely that the firm stops investing. In all, higher r, A^{-1} and γ make it more difficult for firm to build reputation.

[14] In Appendix B, we show that at $X = X^*$, $V'(X+) = V'(X-) = \frac{r(1-A+\gamma)}{2(1-p-q)} g_1(0,1)$. At $X = X^* - \bar{X}_{pq}$, $V'(X+) = \frac{r(1-A+\gamma)}{2(1-p-q)} g_1(0,1) y(X) > 0 = V'(X-)$.

Next, consider the impact of noises p and q on the equilibrium behavior. Corallary 3.1 says that the more noisy the transition (higher p and q) is, the less likely the buyers are to buy (lower $y(X)$). Intuitively, as p and q becomes larger, the incentive in the future is weakened because an one-time no investment causes the reputation to increase with probability p rather than a depreciation of reputation for sure, and an one-time investment decreases the reputation with probability q rather than an increase of reputation for sure. Therefore, the buyers need to compensate the weakening of incentive by increasing the growth rate of $y(X)$. Since $y(X)$ reaches 1 at X^*, a higher growth rate leads to a lower level of $y(X)$ in each state $X < X^*$. Furthermore, larger p and q implies smaller $\bar{X}_{pq}$, which means that it is more likely that the firm ceases to invest. In all, higher p and q make it more difficult for firm to build reputation.

4. Extensions

4.1. **Lower-bound Transition Rules.** The lower bound of the state space $\mathscr{X}_\Delta$ is 0. In *lower-bound transition rules*, the domain of next state X' is either $X + \Delta$ or 0.

(1) If the firm invests, then the probability that the next state $X' = X + \Delta$ is $1 - q$ and the probability that $X' = 0$ is q:

$$P(I, X) = \begin{cases} 1-q & X' = X + \Delta \\ q & X' = 0. \end{cases}$$

(2) If the firm does not invest, then the probability that $X' = 0$ is $1 - p$ and the probability that $X' = X + \Delta$ is p:

$$P(NI, X) = \begin{cases} 1-p & X' = 0 \\ p & X' = X + \Delta. \end{cases}$$

Assumption 4.1: $p + q < 1$.

Assumption 4.2: $\delta > \frac{1-A+\gamma}{1-q-pA}$.

Assumption 4.1 tells us that investing increases reputation stock with a higher probability than not investing: $1 - q > p$, and not investing decreases reputation stock with a higher probability than investing: $1-p > q$. Assumption 4.2 holds for high discount factor δ, small noises p and q, small degree of conflict (large A), and small investment cost γ. Observe that Assumptions 4.1-4.2 allow for a wide range of noises and discount factors.

Theorem 4.1 characterizes the reputation cycle under *lower-bound transition rules*. The equilibrium results work for all fixed time intervals Δ and high discount factors δ. The unique stationary Markov equilibrium is a *non-absorbing equilibrium*, characterized by a reputation cycle with a reputation-building stage and a reputation-exploitation stage. In the former stage, the buyers buy with increasing probability with respect to reputation to provide the firm with the incentives to invest. The firm plays a mixed strategy so that the buyers are indifferent between buying and not buying. The result of a bad outcome is a high probability to ruin reputation to the lowest level. After the ruin, the firm starts over and continues to build reputation. In the later stage, it is a dominant strategy for the buyers to buy. Therefore, the buyers can not reward the firm, so there is no incentive for the firm to build reputation any more. For high discount factors, there is no absorbing state in which firm does not invest and buyers do not buy, thus reputation is a long-run phenomenon.

Theorem 4.1. *Under Assumptions 2.1-2.5, 4.1-4.2 and lower-bound transition rules, the stationary Markov equilibrium is unique and displays a reputation cycle as below:*

(1) Reputation-building stage: $k \leq K-1$. The firm plays mixed strategy $a(k\Delta) = a^*(k\Delta)$ and the buyers play mixed strategy $y(k\Delta) \in (0,1)$ where $y(k\Delta)$ is strictly increasing in k as follows:

(2) Reputation-exploitation stage: $k \geq K$. The firm does not invest for sure and the buyers buy for sure, i.e. $y(k\Delta) = 1$ and $a(k\Delta) = 0$.

Next, we study the limiting equilibrium as $K \to +\infty$ in order to get a clearer analytic solution of buyers' equilibrium behavior and present the comparative statics analysis in Proposition 4.2.[15]

Proposition 4.2. *Under Assumptions 2.1-2.5, 4.1-4.2 and lower-bound transition rules, $y(k)$ is decreasing in $(\delta^{-1}, A^{-1}, \gamma, p, q)$ for any k.*

The stationary Markov equilibrium is a *quasi-absorbing equilibrium* if Assumption 4.2 is violated. Proposition 4.3 tells us that the buyers cannot provide enough incentives for the firm to invest if reputation is low. Furthermore, we can show that as $K \to +\infty$, the number

[15] All the comparative-statics results can be explained by similar arguments as in the *one-step transition rules*

of states in which the firm invests with positive probability is bounded: $K - k^*$ is bounded, which means that the firm loses the incentives to invest at most of the states.

Proposition 4.3. *If Assumption 4.2 does not hold, under Assumptions 2.1-2.5, 4.1 and lower-bound transition rules, the stationary Markov equilibrium has the following features: there exists a unique integer $1 \leq k^* \leq K-1$ such that*
(1) If $0 \leq k \leq k^*$, then $(a(k), y(k)) = (0,0)$.
(2) If $k^*+1 \leq k \leq K-1$, then $a(k) = a^*(k)$ and $y(k) \in (0,1)$.
(3) If $k \geq K$, then $(a(k), y(k)) = (0,1)$.
Moreover, if $K \to +\infty$, then $K - k^*$ is bounded.

4.2. **Augmented One-step Transition Rule.** In previous sections, the buyers have no impact on the accumulation of reputation. In this section, we augment the *one-step transition rules* by allowing the buyers to change the reputation. We analyze the reputation dynamics under the following *augmented one-step transition rule.*

(1) If the buyers do not buy in state X, then the state will remain the same no matter what the firm does.

$$P(X' = X|I, NB, X) = P(X' = X|NI, NB, X) = 1.$$

(2) If the buyers buy in state X, then investing will bring the state one-step up and not investing will bring the state one-step down.

$$P(X' = X + \Delta|I, B, X) = 1,\ P(X' = \max(X - \Delta, 0)|NI, B, X) = 1.$$

Assumption 4.3: $g_1(0,0) = g_1(1,0) = 0$.

Assumption 4.3 says that the firm gets the same payoff 0 if the buyers do not buy, as the firm has no chance of building reputation. Define $K^* = K$ if K is even and $K^* = K+1$ if K is odd. Define $\hat{K} = \lfloor \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{1}{1+\delta} \rfloor + 1$, which determines the existence of an non-absorbing equilibrium. Define $K^{**} \equiv \max(K^* - 2\lfloor \frac{\delta^2}{1-\delta^2} \rfloor - 2, 0)$.

Theorem 4.4. *Under Assumptions 2.1-2.5 and 4.3 and the augmented one-step transition rule, the unique stationary Markov equilibrium is characterized as below:*
(1) $K \leq \hat{K} - 1$. The stationary Markov equilibrium is a non-absorbing equilibrium.
(a) If $0 \leq k \leq K^{**}$, $a(k\Delta) = a^*(k\Delta)$ and $0 < y(k\Delta) < 1$.

(b) *If* $K^{**} \leq k \leq K^* - 1$*, then* $a(k\Delta) = a^*(k\Delta)$ *and* $0 < y(k) < 1$ *in even states, and* $a(k\Delta) = y(k\Delta) = 1$ *in odd states.*

(c) *If* $k \geq K^*$*, then* $(a(k\Delta), y(k\Delta)) = (0, 1)$*.*

(2) $K \geq \hat{K}$*. The stationary Markov equilibrium is a quasi-absorbing equilibrium.*

(a) *If* $0 \leq k \leq K - \hat{K}$*, then* $(a(k\Delta), y(k\Delta)) = (0, 0)$*.*

(b) *If* $K - \hat{K} + 1 \leq k \leq K - 1$*, then* $a(k\Delta) = a^*(k\Delta)$ *and* $0 < y(k\Delta) < 1$*.*

(c) *If* $k \geq K$*, then* $(a(k\Delta), y(k\Delta)) = (0, 1)$*.*

If it is easy for the firm to build reputation to the state in which buyers buy for sure ($K \leq \hat{K} - 1$), then there is a unique *non-absorbing equilibrium* characterized by a reputation-building stage and a reputation-exploitation stage. There is no reputation-absorbing stage in which the firm does not invest and the buyers do not buy if $k \leq k^*$ for some $k^* \leq K - 1$. Indeed, if this "reputation trap" existed, the firm would strictly prefer to invest at $k = k^* + 1$ since reputation could be exploited by investing in the near future ($K \leq \hat{K} - 1$), and not investing would lead to the "reputation trap" with a low payoff 0. Then, the buyers would buy for sure at $k = k^* + 1$ and consequently for all $k > k^* + 1$, the buyers would buy for sure and the firm would invest for sure, contradicting to the fact that firm always exploits the reputation for high enough state. Therefore, there is no reputation-absorbing stage if $K \leq \hat{K} - 1$.

The reputation-building stage is composed of two sub-stages. For lower reputation ($0 \leq k \leq K^{**}$), both the firm and the buyers play mixed strategies. For higher reputation ($K^{**} \leq k \leq K^* - 1$), the incentives for the firm to invest is so high that the firm invests for sure in the odd states, thus the buyers also buy for sure in the odd states. We also show that both players play mixed strategies in even states. The reason is that if there are two consecutive states in which the firm invests for sure, then the firm will invest for sure in the future, a contradiction. In the reputation-exploitation stage ($k \geq K^*$), the firm has no reward of building reputation as the buyers buy for sure. As a result, the reputation moves up and down between $(K^* - 1)\Delta$ and $K^*\Delta$ at which the buyers buy for sure. Therefore, if the buyers are given the maximal power of controlling the reputation, the firm has high incentives to build reputation, and eventually reputation stock cannot escape the two reputation levels at which the buyers buy for sure.

If it is difficult for player 1 to build reputation to the state in which buyers buy for sure ($K \geq \hat{K}$), then there is a unique *quasi-absorbing equilibrium* characterized by three stages: a reputation-absorbing stage, a reputation-building stage and a reputation-exploitation stage. There is a state $(K - \hat{K})\Delta$ at which the future continuation payoff is just not enough for player 1 to build reputation. Any state $k \leq K - \hat{K}$ is an absorbing state, in which the buyers do not buy because he knows that future buyer in the next state will not buy. Therefore, the firm loses the incentive to invest. In the reputation-building stage ($K - \hat{K} + 1 \leq k \leq K - 1$), the buyers will buy with positive probability in an increasing order to provide incentives for the firm to build reputation, and the firm will play a mixed strategy to make the buyers just indifferent between B and NB. In the reputation-exploitation stage ($k \geq K$), the firm has no reward of building reputation since the buyers will buy for sure in all states larger than $K\Delta$.

For a tractability solution, we consider an analytic solution of the limiting equilibrium as $\Delta \to 0$ as follows: define $\hat{X} = \frac{1}{r}\frac{1+A}{1-A}$.

(1) $X^* \leq \hat{X}$. The equilibrium is a *non-absorbing equilibrium*.

(a) If $0 \leq X \leq \max(X^* - \frac{1}{r}, 0)$, then

$$(a(X), y(X)) = \begin{cases} (a^*(X), \frac{(1+A)-r(1-A)(X^*-X)}{2A}) & X = \lim_{\Delta\to 0}(2k+1)\Delta \\ (a^*(X), \frac{(1+A)-r(1-A)(X^*-X)}{2}) & X = \lim_{\Delta\to 0} 2k\Delta. \end{cases}$$

(b) If $\max(X^* - \frac{1}{r}, 0) < X < X^*$, then

$$(a(X), y(X)) = \begin{cases} (1, 1) & X = \lim_{\Delta\to 0}(2k+1)\Delta \\ (a^*(X), \frac{1+A-r(1-A)(X^*-X)}{1+A+r(1-A)(X^*-X)}) & X = \lim_{\Delta\to 0} 2k\Delta. \end{cases}$$

(c) If $X \geq X^*$, $(a(X), y(X)) = (0, 1)$.

(2) $X^* > \hat{X}$. The equilibrium is a *quasi-absorbing equilibrium*.

(a) $0 \leq X \leq X^* - \hat{X}$, $a(X) = y(X) = 0$.

(b) $X^* - \hat{X} < X \leq X^*$, $(a(X), y(X)) = (a^*(X), 1 - \frac{1-A}{1+A} r(X^* - X))$.

(c) $X \geq X^*$, $(a(X), y(X)) = (0, 1)$.

(3) $\hat{X}$ is decreasing in (A^{-1}, r), and $y(X)$ is non-increasing in (A^{-1}, r) for any X.

Qualitatively similar to that in the *one-step transition rules*, Proposition 4.5 characterizes the limiting equilibrium under *augmented one-step transition rule* and finds the necessary and sufficient condition for the existence of absorbing states and how the equilibrium behavior is influenced by the changes of parameters.

Proposition 4.5. *Under Assumptions 2.1-2.5, 4.3 and the augmented one-step transition rule, the unique stationary Markov equilibrium in the limit as $\Delta \to 0$ is characterized as above.*

4.3. **Multiple Investment Levels.** In the previous sections, we assume that the firm has only two choices: investment I and no investment NI. In this section, we relax the assumption that there is only one investment choice. Instead, there are n investment choices: $\{I_i\}_{i=1}^n$ and a choice of no investment $NI \equiv I_0$. Assume that in the next period, reputation can only go one-step up or down. If the buyers choose $B(NB)$ and the firm chooses I_i, then denote $g_1(I_i, B)$ $(g_1(I_i, NB))$ as firm's stage game payoff and denote $g_2(I_i, B, X)$ $(g_2(I_i, NB, X))$ as each buyer's stage game payoff if the state is X.

Assumption 4.4: $g_1(I_i, B) > g_1(I_j, B)$, $g_1(I_i, NB) \geq g_1(I_j, NB)$ for any $i < j$.

Assumption 4.5: $c_i \equiv g_1(I_0, B) - g_1(I_i, B) > g_1(I_0, NB) - g_1(I_i, NB)$ for any $1 \leq i \leq n$.

Assumption 4.6: $g_2(I_i, B, X) > g_2(I_i, NB)$ for any $1 \leq i \leq n$. $g_2(I_i, NB) = g_2(I_j, NB)$ for any $0 \leq i, j \leq n$.

Assumption 4.7: $g_2(I_i, B, X)$ is strictly increasing in X.

Assumption 4.8: There is X^* such that if $X \geq X^*$ then $g_2(I_0, B, X) \geq g_2(I_0, NB)$, otherwise $g_2(I_0, B, X) < g_2(I_0, NB)$.

Assumption 4.9: $g_1(I_i, NB) = g_1(I_j, NB) = 0$ for any $0 \leq i,\ j \leq n$.

Assumptions 4.4-4.8 is the same as Assumptions 2.1-2.5 if we restrict the model to two choices I_i and I_0. Assumption 4.8 tells us that if $X \geq X^*$, it is a dominant strategy for the buyers to play B. It is reasonable to assume that the buyers will buy the product for sure independent of the firm's current behavior because the firm has done good enough in the past. If $X < X^*$, then there is a mixed strategy $a_i^*(X) \in (0,1)$ of playing I_i and $1 - a_i^*(X)$ of playing I_0 to make the buyers be indifferent between B and NB. Assumption 4.9 is a simplifying assumption. If the buyers choose not to buy the product, then any investment level I_i will bring the same payoff 0 to the firm.

Next, we focus on *one-step transition rules* as follows:

(1) If the firm invests at the level of I_i, then the probability that the next state $X' = X + \Delta$ is $1 - q_i$ and the probability that $X' = \max\{X - \Delta, 0\}$ is q_i:

$$P(X'|I_i) = \begin{cases} 1 - q_i & X' = X + \Delta \\ q_i & X' = \max(X - \Delta, 0). \end{cases}$$

(2) If the firm does not invest, then the probability that $X' = \max(X - \Delta, 0)$ is $1 - p$ and the probability that $X' = X + \Delta$ is p:

$$P(X'|I_0) = \begin{cases} p & X' = X + \Delta \\ 1 - p & X' = \max(X - \Delta, 0). \end{cases}$$

Without loss of generality, assume that $c_i > c_j$ for $i > j$. Assumption 4.10 tells us that an investment with larger cost leads to a higher probability of one-step increase of reputation in the next period.

Assumption 4.10 : $q_i < q_j$ for $i > j$.

Denote $i^* = \arg\min_{i \geq 1}\{\frac{c_i}{q_0 - q_i}\}$. Therefore, c_{i^*} is the most "efficient" investment level in the sense that the marginal cost is minimized relative to marginal benefit. Define

$$A = \frac{g_1(I_{i^*}, B)}{g_1(I_0, B)},\ A_{i^*} = \frac{(1-p)A - q_{i^*}}{1 - q_{i^*} - Ap}.$$

Assumption 4.11: $A > \frac{q_i^*}{1-p}$ for any $1 \leq i \leq n$.

Assumption 4.11 guarantees that $A_{i^*} \in (0, 1)$. The number A captures the investment cost of I_{i^*}: higher A means lower investment cost of I_{i^*}. $\frac{q_i^*}{1-p}$ captures the benefit of I_{i^*}: lower $\frac{q_i^*}{1-p}$ means higher benefit of I_{i^*}. Therefore, Assumption 4.11 tells us that the cost of investing cannot be too high relative to the benefit of investing.

Theorem 4.3 constructs a stationary Markov equilibrium in which the firm only mixes between the "efficient" investment level I_{i^*} and not investing I_0 and the buyers play mixed strategies in the reputation-building stage ($X < X^*$). In the reputation-exploitation stage ($X \geq X^*$), the firm does not invest and the buyers buy. However, this may not be the only stationary Markov equilibrium if we allow the buyers to choose pure strategies in the reputation-building stage.

Theorem 4.6. *Under Assumptions 4.4-4.11 and one-step transition rules, there is a stationary Markov equilibrium as below: there exists an integer* $M(p, q_i^*) > 0$ *for each* p *and* q^* *such that*

(1) Reputation-building stage: $M(p, q_i^*) \leq k \leq K-1$. *The firm plays* I_{i^*} *with probability* $a_{i^*}^*(k\Delta)$ *and plays* I_0 *with probability* $1 - a_{i^*}^*(k\Delta)$. *The buyers also play mixed strategy* $y(k\Delta) \in (0,1)$, *which is characterized by a second-order difference equation:*

$$y((k+1)\Delta) = \frac{1}{\delta}(1 - A_{i^*})y(k\Delta) + A_{i^*}y((k-1)\Delta) \; \forall 1 \leq k \leq K-2.$$

(2) Reputation-exploitation stage: $k \geq K$. *The firm does not invest and the buyers do not buy, i.e.* $y(k\Delta) = 1$ *and* $a(k\Delta) = 0$.

5. Conclusion

In this paper, we study reputation dynamics in a setting of stochastic games in which reputation is modeled as a state variable, rather than a belief as in the traditional reputation literature. Under a class of transition rules, the unique stationary Markov equilibrium is characterized by a reputation-building phase, a reputation-exploitation phase and a possible reputation-absorbing stage. Under certain conditions, there is no absorbing state and reputation is a long-run phenomenon, which moves cyclically between the reputation-building stage and the reputation-exploitation stage. Therefore, the paper provides a new rationale for permanent reputations, in line with the recent experience of Toyota with the recalls. Furthermore, the result is robust under different transition rules including the case in which the buyers also have the power of controlling the evolution of reputation.

Based on this paper, there are several extensions, namely, non-*submodularity*, competition and multidimensional reputation. This paper assumes that the firm's payoff is subject to *submodularity*, which is common in the reputation literature (Liu, 2011; Liu and Skrzypacz, 2014; Phelan, 2006). Intuitively, *submodularity* reflects situations where the players have conflicting interests. There are two other cases of interest: *supermodularity* (common interests) and independent interests. In the online appendix, we analyze the independent interest case in which the investment cost is a constant, independently of the buyers' choices, and shows that the qualitative features are similar to the *submodularity* case in this paper. The common interests case in which the firm's payoffs display *supermodularity* is the object of future research.

Faced with competition, a firm builds reputation because it wants to differentiate its product from other firms. Therefore, we can study the industry dynamics when there are multiple firms in the market. It is interesting to investigate firms' exit and entry decisions and the stationary distribution of reputation in a steady-state equilibrium. As a first step, Huangfu (2014a) studies a model with two long-run firms competing for a sequence of short-run buyers in each period. Since the buyers' choices only depend on the relative reputation of the two firms, a natural sufficient statistic is reputation difference of the two firms. It would be interesting to know whether the leading firm perpetually enlarge the leadership or the follower eventually catch up. Under certain circumstances, Huangfu (2014a) shows that the latter is true: the leader has less incentive to invest than the follower. As a result, there are reputation cycles in which the leadership changes over time.

A firm may have multidimensional reputation to manage. For example, an automobile company may have multiple sub-brands to sell or may have only a brand to sell but buyers care about different dimensions of the car quality: performance, reliability or appearance. Therefore, it is useful to study how a firm allocates its resource in order to optimally manage its multidimensional reputation. Huangfu (2014b) establishes that in a model of two dimensions of reputation, a firm will focus on a certain dimension with relatively higher reputation and build this dimension to a very high level and then starts to allocate resource to a new dimension because a low effort is enough to maintain reputation of the old dimension.

Appendix A. Proofs of Theorem 3.1

Outline of the Proof of Theorem 3.1 if $p = q = 0$.

(1) Lemma A.1 shows that if the buyer does not buy at state k ($y_k = 0$), then the buyer will not buy at any smaller state ($y_i = 0\ \forall\ 0 \le i \le k$). Therefore, any equilibrium can be divided into two kinds as follows: (i) $y_i > 0$ for any $i \ge 0$; (ii) there exists $k^* \ge 1$ such that $y_i = 0$ if and only if $0 \le i \le k^* - 1$.

(2) Lemma A.2 shows that it is impossible that the buyers buy for sure for two consecutive states for $k \le K$. Otherwise, the firm will invest for sure in all future states, which is impossible because such incentives cannot be provided by the buyers.

(3) Consider the *non-absorbing equilibrium*: $y_0 > 0$. By Lemma A.1, $y_i > 0$ for any $i \ge 0$.

   (a) For small Δ, show that the firm does not invest in state $k \ge K$.

   (b) If the firm does not invest in state K, then by Lemma A.2, we use backward induction to show that $y_i \in (0,1)$ for any $2 \leq i \leq K-1$. By solving a second-order difference equation, we show the uniqueness of the non-absorbing equilibrium.
   (c) Use the solution of $\{y_i\}_{i=0}^{K-1}$ to find the necessary condition under which $y_0 > 0$: $K \leq \bar{K}_{00}$.

(4) Consider the *quasi-absorbing equilibrium*: $y_0 = 0$. By Lemma A.1, there exists $k^* \geq 1$ such that $y_i = 0$ if and only if $0 \leq i \leq k^* - 1$.
   (a) For $k^* \leq k \leq K$, we use the same method as in Step 3 to show the uniqueness of the *quasi-absorbing equilibrium* and characterize it.
   (b) Show that $k^* = K - \bar{K}_{00}$, thus the necessary condition for the existence of a *quasi-absorbing equilibrium* is $K \geq \bar{K}_{00} + 1$.

(5) If $K \leq \bar{K}_{00}$, then by Step 3(c), the equilibrium satisfies $y_0 = 0$ and is the unique *quasi-absorbing equilibrium* characterized in Step 4. If $K \leq \bar{K}_{00}$, then by Step 4(b), the equilibrium satisfies $y_0 > 0$ and is the unique *non-absorbing equilibrium* characterized in Step 3.

In this section, V_k and y_k denote $V(k\Delta)$ and $y(k\Delta)$. For notational convenience, we use $g_1(a, y)$ instead of $(1-\delta)g_1(a, y)$.

Lemma A.1: For any $p \geq 0$ and $q \geq 0$, if $y_{k+1} = 0$, then $y_i = 0$ for all $0 \leq i \leq k$.

Proof. **Step 1**: $y_0 < 1$.

If $y_1 = 1$, then $V_0 > g_1(0,1) + \delta(pV_1 + (1-p)V_0) > g_1(0,1) + \delta V_0$. Therefore, $V_0 > \frac{g_1(0,1)}{1-\delta}$, a contradiction to the fact that $g_1(0,1)$ is the maximal stage-game payoff.

Step 2: If $y_1 = 0$, then $y_0 = 0$.

Assume by contradiction that $y_0 > 0$. By Step 1, $0 < y_0 < 1$. Therefore, $V_0 = g_1(0, y_0) + \delta(pV_1 + (1-p)V_0) = g_1(1, y_0) + \delta(qV_0 + (1-q)V_1)$ and thus $V_1 - V_0 = \frac{1}{\delta(1-p-q)}(g_1(0, y_0) - g_1(1, y_0)) > 0$.

$y_1 = 0$ implies that $V_1 = g_1(0,0) + \delta(pV_2 + (1-p)V_0) = g_1(1,0) + \delta(qV_0 + (1-q)V_2)$ and $V_2 - V_0 = \frac{1}{\delta(1-p-q)}(g_1(0,0) - g_1(1,0)) < \frac{1}{\delta(1-p-q)}(g_1(0, y_0) - g_1(1, y_0)) = V_1 - V_0$. Therefore, $V_2 < V_1$. Then, $V_0 = g_1(0, y_0) + \delta(pV_1 + (1-p)V_0) > g_1(0,0) + \delta(pV_2 + (1-p)V_0) = V_1$, a contradiction to $V_1 > V_0$.

Step 3: If $y_{k+1} = 0$ for $k \geq 1$, then $y_k = 0$.

Assume by contradiction that $y_k > 0$. Show that for any $1 \le i \le k$, $y_{k-i} = 1$ and $V_{k-i+1} - V_{k-i} \le V_{k-i} - V_{k-i-1}$.

First, check the case that $i = 1$. By $y_{k+1} = 0$, $V_{k+1} - V_{k-1} \le \delta(p(V_{k+2} - V_k) + (1 - p)(V_k - V_{k-2}))$. As $V_{k+1} - V_{k-1} = \frac{1}{\delta(1-p-q)}(g_1(0, y_k) - g_1(1, y_k)) > V_{k+2} - V_k$, then $V_{k+2} - V_k < V_{k+1} - V_{k-1} < V_k - V_{k-2}$. If $y_{k-1} < 1$, then $0 < y_k < 1$. We can show that $y_{k-1} < 1$, $0 < y_k < 1$ and $y_{k+1} = 0$ imply that $\frac{\gamma}{1-A} > (\frac{1-A_{pq}}{\delta} + A_{pq})\frac{\gamma}{1-A}$, a contradiction. Therefore, $y_{k-1} = 1$. $y_k \le 1 = y_{k-1}$ implies that $V_k - V_{k-1} \le \delta(q(V_{k-1} - V_{k-2}) + (1 - q)(V_{k+1} - V_k))$, thus $V_k - V_{k-1} < V_{k-1} - V_{k-2}$.

Assume by induction that for any $1 \le j \le i-1$, $y_{k-j} = 1$ and $V_{k-j+1} - V_{k-j} \le V_{k-j} - V_{k-j-1}$. Now, show that it is true for $j = i$.

By $y_{k+1} = 0$, $V_{k+1} - V_{k-i} \le \delta(p(V_{k+2} - V_{k-i+1}) + (1 - p)(V_k - V_{k-i-1})) = \delta(p(V_{k+2} - V_k) + (1 - p)(V_{k-i+1} - V_{k-i-1}) + V_k - V_{k-i+1})$. By induction hypothesis, $V_k - V_{k-1} < V_{k-i+1} - V_{k-i}$. Then, we can show that $V_{k+1} - V_{k-1} \le \delta(p(V_{k+2} - V_k) + (1 - p)(V_{k-i+1} - V_{k-i-1}))$. By induction hypothesis, $V_{k+2} - V_k < V_{k-i+1} - V_{k-i-1}$, then $V_{k+1} - V_{k-1} < V_{k-i+1} - V_{k-i-1}$. In all, $y_{k-i} = 1$. Furthermore, $V_{k-i+1} - V_{k-i} \le \delta(q(V_{k-i} - V_{k-i-1}) + (1 - q)(V_{k-i+2} - V_{k-i+1}))$ and thus $V_{k-i+1} - V_{k-i} < V_{k-i} - V_{k-i-1}$. Therefore, $y_0 = 1$, a contradiction to Step 1.

In all, we have shown that $y_k = 0$.

Step 4: If $y_{k+1} = 0$ for $k \ge 1$, then $y_i = 0$ for all $0 \le i \le k$.

Use the same argument as in Step 3, we can show by induction that if $y_{k+1} = 0$ for $k \ge 1$, then $y_i = 0$ for all $1 \le i \le k$. By Step 2, if $y_1 = 0$, then $y_0 = 0$.

□

Lemma A.2: If $p = q = 0$, then it is impossible that the buyers buy the product for sure at two consecutive states: $y_k = y_{k+1} = 1$ for any $1 \le k \le K - 2$.

Proof. If $y_k = y_{k+1} = 1$ for some $0 \le k \le K - 2$, then $V_k = g_1(1, 1) + \delta V_{k+1} \ge g_1(0, 1) + \delta V_{k-1}$ and $V_{k+1} = g_1(1, 1) + \delta V_{k+2} \ge g_1(0, 1) + \delta V_k$. Then, $V_{k+1} - V_k = \delta(V_{k+2} - V_{k+1}) < V_{k+2} - V_{k+1}$. Therefore, $V_{k+2} \ge g_1(0, 1) + \delta V_k + (V_{k+2} - V_{k+1}) > g_1(0, 1) + \delta V_k + (V_{k+1} - V_k) = g_1(0, 1) + \delta V_{k+1}$. Therefore, the firm strictly prefers I to NI at period $k + 2$. Then, $V_{k+2} = g_1(1, 1) + \delta V_{k+3}$. By induction, we can show that for all $t \ge k$, $V_t = g_1(1, 1) + \delta V_{t+1}$ $\forall t \ge k$.

Since $\{V_t\}_{t\geq k}$ is a strictly increasing and bounded sequence, there is a limit V^* such that $V^* = g_1(1,1) + \delta V^*$. Therefore, $V_{t+1} < V^* = \frac{g_1(1,1)}{1-\delta}$ for any $t \geq k$. However, $V_{t+1} > V_t = g_1(1,1) + \delta V_{t+1}$ and hence $V_{t+1} > \frac{g_1(1,1)}{1-\delta}$, a contradiction.

□

Lemma A.3: If $p = q = 0$, then (1) If $0 < y_k < 1$ and $0 < y_{k+1} < 1$, then $z_{k+1} = \frac{1}{\delta}(1-A)z_k + Az_{k-1}$; (2) If $y_{k+1} = 1$, then $z_{k+1} \leq \frac{1}{\delta}(1-A)z_k + Az_{k-1}$; (3) If $y_{k+1} = 1$ and $V_{k+2} = g_1(0, y_{k+2}) + \delta V_{k+1}$, then $z_{k+2} \geq \frac{1}{\delta}(1-A)z_{k+1} + Az_k$.

Proof. Because $y_k > 0$ for all $0 \leq k \leq K-1$, $V_k = g_1(1, y_k) + \delta V_{k+1}$ for all $0 \leq k \leq K-1$. Define $z_k = y_k + \frac{\gamma}{1-A}$.

(1) $0 < y_k < 1$ and $0 < y_{k+1} < 1$. We can show that $(1-\delta^2)V_k = g_1(0, y_k) + \delta g_1(1, y_{k-1}) = g_1(1, y_k) + \delta g_1(0, y_{k+1})$. Therefore, $z_{k+1} = \frac{1}{\delta}(1-A)z_k + Az_{k-1}$.

(2) $y_{k+1} = 1$. By Lemma A.2, we have $0 < y_k < 1$. Then, $(1-\delta^2)V_k = g_1(0, y_k) + \delta g_1(1, y_{k-1}) \geq g_1(1, y_k) + \delta g_1(0, y_{k+1})$. Therefore, $z_{k+1} \leq \frac{1}{\delta}(1-A)z_k + Az_{k-1}$.

(3) $y_{k+1} = 1$ and $V_{k+2} = g_1(0, y_{k+2}) + \delta V_{k+1}$. By Lemma A.2, we have $0 < y_k < 1$. Then, $(1-\delta^2)V_{k+1} = g_1(1, y_{k+1}) + \delta g_1(0, y_{k+2}) \geq g_1(0, y_{k+1}) + \delta g_1(1, y_k)$. Therefore, $z_{k+2} \geq \frac{1}{\delta}(1-A)z_{k+1} + Az_k$.

□

Lemma A.4: Under $p = q = 0$, if $K > 3 + \frac{\log \frac{\epsilon-1}{A}}{\log \frac{A}{\epsilon}}$, then the firm does not invest in state K and $0 < y_k < 1$ for all $2 \leq k \leq K-1$. $\epsilon = \frac{1}{2\delta}(1 - A + \sqrt{(1-A)^2 + 4A\delta^2})$.

Proof. **Step 1:** If the firm strictly prefers to play NI in state K and $y_{K-1} < 1$, then $0 < y_k < 1$ for all $2 \leq k \leq K-2$.

By lemma A.3(3), $z_K \geq \frac{1}{\delta}(1-A)z_{K-1} + Az_{K-2}$. If $y_{K-2} = 1$, then $z_{K-2} = 1 + \frac{\gamma}{1-A}$ and $z_{K-1} \leq \delta(1 + \frac{\gamma}{1-A})$. Since $y_{K-2} = 1$, $y_{K-3} < 1$ by Lemma A.2. By Lemma A.3(2) , $z_{K-2} \leq \frac{1}{\delta}(1-A)z_{K-3} + Az_{K-4} \leq \frac{1}{\delta}(1-A)z_{K-3} + A(1 + \frac{\gamma}{1-A})$. Then, $z_{K-3} \geq \delta(1 + \frac{\gamma}{1-A})$. By Lemma A.3(3), $z_{K-1} \geq \frac{1}{\delta}(1-A)z_{K-2} + Az_{K-3}$, then $z_{K-1} > z_{K-3} \geq \delta(1 + \frac{\gamma}{1-A})$, a contradiction to $z_{K-1} \leq \delta(1 + \frac{\gamma}{1-A})$. In all, we have shown that $y_{K-2} < 1$.

Show that $0 < y_k < 1$ for all $2 \leq k \leq K-2$ by induction. Assume $y_t < 1$ for all $t \geq k$. Assume $y_{k-1} = 1$, then $y_{k-2} < 1$. By Lemma A.3(2), $z_{k-1} \leq \frac{1}{\delta}(1-A_p)z_{k-2} + A_p z_{k-3} \leq \frac{1}{\delta}(1-A)z_{k-2} + A(1 + \frac{\gamma}{1-A})$. Then, $z_{k-2} \geq \delta(1 + \frac{\gamma}{1-A})$. By Lemma A.3(3), $z_k \geq \frac{1}{\delta}(1-A)z_{k-1} + Az_{k-2}$, then $z_k > z_{k-2} \geq \delta(1+\frac{\gamma}{1-A})$. Therefore, $z_{k+1} = \frac{1}{\delta}(1-A)z_k + Az_{k-1} > 1+\frac{\gamma}{1-A}$, a contradiction. In all, we have show that $0 < y_k < 1$ for all $2 \leq k \leq K-2$.

Step 2 : The firm strictly prefers to play NI in state K.

Assume that the firm weakly prefers to play C in state K. By the same logic of Lemma A.2, the firm strictly prefers to play NI in state $K+1$. By Lemma A.3(3), $z_{K+1} \geq \frac{1}{\delta}(1-A)z_K + Az_{K-1}$. Then, $z_{K-1} \leq \frac{1-\frac{1}{\delta}(1-A)}{A}(1+\frac{\gamma}{1-A}) = (\frac{1}{\epsilon} - \frac{\epsilon-1}{A})(1+\frac{\gamma}{1-A})$.

Figure out the lower bound of z_{K-1}. If $y_{K-2} < 1$, then by the same argument of Step 1, we have $0 < y_k < 1$ for all $2 \leq k \leq K-2$. Then, $z_K - \epsilon z_{K-1} \leq (-\frac{A}{\epsilon})^{K-2}(z_2 - \epsilon z_1)$. Therefore, $z_{K-1} \geq (\frac{1}{\epsilon} - (\frac{A}{\epsilon})^{K-2})(1+\frac{\gamma}{1-A})$. If $y_{K-2} = 1$, then by Lemma A.2, we have $y_{K-3} < 1$. By the same argument of Step 1, $0 < y_k < 1$ for all $2 \leq k \leq K-3$. By Lemma A.3(3), $z_{K-1} \geq \frac{1}{\delta}(1-A)z_{K-2} + Az_{K-3}$. Therefore, $z_{K-1} - \epsilon z_{K-2} \geq (-\frac{A}{\epsilon})^{K-3}(z_2 - \epsilon z_1)$. As $z_K - \epsilon z_{K-1} \leq (-\frac{A}{\epsilon})(z_{K-1} - \epsilon z_{K-2}) \leq (-\frac{A}{\epsilon})^{K-2}(z_2 - \epsilon z_1)$, then $z_{K-1} \geq (\frac{1}{\epsilon} - (\frac{A}{\epsilon})^{K-2})(1+\frac{\gamma}{1-A})$.

The upper and lower bound of z_{K-1} implies that $\frac{\epsilon-1}{A} \leq (\frac{A}{\epsilon})^{K-2}$, a contradiction to $K > 3 + \frac{\log\frac{\epsilon-1}{A}}{\log\frac{A}{\epsilon}}$. In all, the firm strictly prefers NI in state K.

Step 3 : $y_{K-1} < 1$.

Assume that $y_{K-1} = 1$. We have shown in Step 2 that the firm strictly prefer NI in state K. Therefore, $z_K \geq \frac{1}{\delta}(1-A_p)z_{K-1} + Az_{K-2}$. Then, $z_{K-2} \leq (\frac{1-\frac{1}{\delta}(1-A)}{A})(1+\frac{\gamma}{1-A}) = (\frac{1}{\epsilon} - \frac{\epsilon-1}{A})(1+\frac{\gamma}{1-A})$.

Figure out the lower bound of z_{K-2}. If $y_{K-3} < 1$, then by the same argument of Step 1, $0 < y_k < 1$ for all $2 \leq k \leq K-3$. We can estimate z_{K-2}: $z_{K-1} - \epsilon z_{K-2} \leq (-\frac{A_p}{\epsilon})^{K-3}(z_2 - \epsilon z_1)$ and thus $z_{K-2} \geq (\frac{1}{\epsilon} - (\frac{A}{\epsilon})^{K-3})(1+\frac{\gamma}{1-A})$.

If $y_{K-3} = 1$, then by Lemma A.2, we have $y_{K-4} < 1$. By the same argument of Step 1, $0 < y_k < 1$ for all $2 \leq k \leq K-4$. Because $z_{K-2} \geq \frac{1}{\delta}(1-A)z_{K-3} + Az_{K-4}$, $z_{K-2} - \epsilon z_{K-3} \geq (-\frac{A}{\epsilon})^{K-4}(z_2 - \epsilon z_1)$. Then, $z_{K-1} - \epsilon z_{K-2} \leq (-\frac{A}{\epsilon})(z_{K-2} - \epsilon z_{K-3}) \leq (-\frac{A}{\epsilon})^{K-3}(z_2 - \epsilon z_1)$. In all, $z_{K-2} \geq (\frac{1}{\epsilon} - (\frac{A}{\epsilon})^{K-3})(1+\frac{\gamma}{1-A})$.

The upper and lower bound of z_{K-1} implies that $\frac{\epsilon-1}{A} \leq (\frac{A}{\epsilon})^{K-3}$, a contradiction to $K > 3 + \frac{\log\frac{\epsilon-1}{A}}{\log\frac{A}{\epsilon}}$. In all, $y_{K-1} < 1$.

Step 4 : The firm strictly prefers NI in state $t > K$.

Assume that the firm weakly prefers C in state $K+i$ where $i \geq 1$. By the same argument of Lemma A.3, the firm strictly prefers NI in state $K+i+1$. Therefore, we can show that $(1-\delta^2)V_{K+i} = g_1(1,1)+\delta g_1(0,1) \geq g_1(0,1)+\delta g_1(1,1)$, a contradiction to $g_1(0,1) > g_1(1,1)$. □

Proof of Theorem 3.1 if $p = q = 0$:

Proof. **Step 1:** Show the uniqueness of *non-absorbing equilibrium*: $y_0 > 0$ and characterize it.

Firstly, show that there exists some $\bar{\Delta}_{00} > 0$ such that if $\Delta < \bar{\Delta}_{00}$, then $K > 3 + \log\frac{\epsilon-1}{A}/\log\frac{A}{\epsilon}$.

By the definition of ϵ, we can show that $\lim_{\Delta\to 0}\epsilon e^{-r\Delta} = 1$. Therefore, $\lim_{\Delta\to 0}\Delta\log\frac{e^{r\Delta}-1}{A}/\log A = \lim_{\Delta\to 0}\Delta\log\frac{r\Delta}{A}/\log A = 0$. Furthermore, $\lim_{\Delta\to 0}(K-3)\Delta = X^* > 0$. In all, for Δ small enough, $K > 3+\log\frac{\epsilon-1}{A}/\log\frac{A}{\epsilon}$.

By Lemma A.4, if $K > 3+\log\frac{\epsilon-1}{A}/\log\frac{A}{\epsilon}$, the firm strictly prefers to play NI in state $k \geq K$ and $0 < y_k < 1$ for all $2 \leq k \leq K-1$. Furthermore, the buyers buy for sure in state $k \geq K$ and play mixed strategy $a^*(k\Delta)$ in the state $2 \leq k \leq K-1$.

In order to solve for y_k for any $1 \leq k \leq K-1$, there are two cases for us to consider: $y_1 = 1$ and $y_1 < 1$.

Case 1: $y_1 < 1$.

By lemma A.3(1), for any $1 \leq k \leq K-1$, $z_{k+1} = \frac{1}{\delta}(1-A)z_k + Az_{k-1}$. Furthermore, $z_1 = (\frac{1}{\delta}(1-A)+1)z_0$. By $z_K = 1+\frac{\gamma}{1-A}$, the solution is

$$z_k = \frac{(1+\epsilon)\epsilon^k - (1-\frac{A}{\epsilon})(-\frac{A}{\epsilon})^k}{(1+\epsilon)\epsilon^K - (1-\frac{A}{\epsilon})(-\frac{A}{\epsilon})^K}(1+\frac{\gamma}{1-A})\ \forall 0 \leq k \leq K-1.$$

In order to satisfy $y_1 < 1$, we need $z_1 < 1+\frac{\gamma}{1-A}$. Therefore, $\frac{(\epsilon+\frac{A}{\epsilon})(\epsilon+1-\frac{A}{\epsilon})}{(1+\epsilon)\epsilon^K-(1-\frac{A}{\epsilon})(-\frac{A}{\epsilon})^K} < 1$.

Case 2: $y_1 = 1$.

If $\frac{(\epsilon+\frac{A}{\epsilon})(\epsilon+1-\frac{A}{\epsilon})}{(1+\epsilon)\epsilon^K-(1-\frac{A}{\epsilon})(-\frac{A}{\epsilon})^K} > 1$, then there is no solution as in Case 1, otherwise $y_1 > 1$, a contradiction. The only possible case is that the firm strictly prefers I in state 1. Then, for any $2 \leq k \leq K-1$, $z_{k+1} = \frac{1}{\delta}(1-A)z_k + Az_{k-1}\ \forall 2 \leq k \leq K-1$. Furthermore, $z_1 = 1+\frac{\gamma}{1-A}$.

By $z_K = 1+\frac{\gamma}{1-A}$, the solution is

$$z_k = (\epsilon^{k-1} + (1-\epsilon^{K-1})\frac{\epsilon^{k-1}-(-\frac{A}{\epsilon})^{k-1}}{\epsilon^{K-1}-(-\frac{A}{\epsilon})^{K-1}})(1+\frac{\gamma}{1-A})\ \forall 1 \leq k \leq K-1.$$

z_0 can be solved by $z_2 = \frac{1-A+\delta+A\delta^2}{\delta^2} z_0 - \frac{A}{\delta} z_1$, which comes from the firm's optimality condition at state 0.

Step 2: Show that the necessary condition for the existence of a *non-absorbing equilibrium* is $K \leq \bar{K}_{00}$, where $\bar{K}$ is the largest integer to satisfy $(1+\epsilon)\epsilon^K - (1-\frac{A}{\epsilon})(-\frac{A}{\epsilon})^K < \frac{1-A+\gamma}{\gamma}(\epsilon + \frac{A}{\epsilon})$.

Check the condition to guarantee $y_0 > 0$: $z_0 = \frac{\epsilon + \frac{A}{\epsilon}}{(1+\epsilon)\epsilon^K - (1-\frac{A}{\epsilon})(-\frac{A}{\epsilon})^K}(1 + \frac{\gamma}{1-A}) > \frac{\gamma}{1-A}$. Therefore, $(1+\epsilon)\epsilon^K - (1-\frac{A}{\epsilon})(-\frac{A}{\epsilon})^K < \frac{1-A+\gamma}{\gamma}(\epsilon + \frac{A}{\epsilon})$. Because the LHS is increasing in K if $K > 3 + \log\frac{\epsilon-1}{A} / \log\frac{A}{\epsilon}$, then there is a cutoff $\bar{K}_{00}$, which is the largest integer to satisfy the above inequality. If $K \leq \bar{K}_{00}$, then the above inequality holds. If $K \geq \bar{K}_{00} + 1$, then the above inequality does not hold.

In all, we need $K \leq \bar{K}_{00}$ to guarantee the existence of a *non-absorbing equilibrium.*

Step 3: Show the uniqueness of a *quasi-absorbing equilibrium*: $y_0 = 0$ and characterize it.

Define $n \geq 1$ as the smallest state such that $y_n > 0$. Therefore, $y_k = 0$ for all $0 \leq k \leq n-1$. Then, $V_k = \frac{g_1(0,0)}{1-\delta}$ for all $0 \leq k \leq n-1$. Moreover, $V_n = g_1(0, y_n) + \delta V_{n-1} = g_1(1, y_n) + \delta V_{n+1}$ and $V_{n+1} = g_1(0, y_{n+1}) + \delta V_n = g_1(1, y_{n+1}) + \delta V_{n+2}$. Therefore, $z_{n+1} = (\frac{1-A}{\delta} + 1)z_n + (1 + \delta)(\frac{\gamma}{1-A} - z_n)$.

Combined with $z_{k+2} = \frac{1}{\delta}(1-A)z_{k+1} + Az_k$ for $n \leq k \leq K-2$ and $z_K = 1 + \frac{\gamma}{1-A}$, there is a unique solution z_k for $n \leq k \leq K-1$.

Step 4: Show that the necessary condition for the existence of a *quasi-absorbing equilibrium* is $K \geq \bar{K}_{00} + 1$.

Show that $n = K - \bar{K}_{00}$. Define $f(n) \equiv \frac{\epsilon + \frac{A}{\epsilon}}{(1+\epsilon)\epsilon^{K-n} - (1-\frac{A}{\epsilon})(-\frac{A}{\epsilon})^{K-n}}(1 + \frac{\gamma}{1-A})$. We can show that $(1-\theta)(z_n - \frac{\gamma}{1-A}) = f(n) - \frac{\gamma}{1-A}$. By the firm's optimality condition at state $n-1$, it is true that $z_n - \frac{\gamma}{1-A} < \frac{\gamma}{\delta}$. Furthermore, it is trivial that $z_n - \frac{\gamma}{1-A} > 0$. Therefore, $\frac{\gamma}{1-A} < f(n) < \frac{\gamma}{1-A} + \frac{(1-\theta)\gamma}{\delta}$. Moreover, $f(n-1) = \frac{1}{1+\frac{(1-A)(1-\theta)}{\delta}} f(n) < \frac{1}{1+\frac{(1-A)(1-\theta)}{\delta}}(\frac{\gamma}{1-A} + \frac{(1-\theta)\gamma}{\delta}) = \frac{\gamma}{1-A}$. In all, we have shown that $f(n-1) < \frac{\gamma}{1-A} < f(n)$. By the definition of $\bar{K}_{00}$, $f(n-1) < f(K - \bar{K}_{00}) < f(n)$. Therefore, $n = K - \bar{K}_{00}$, thus $K \geq \bar{K}_{00} + 1$.

Step 5: By Step 2, if $K \geq \bar{K} + 1$, then the equilibrium is a *quasi-absorbing equilibrium*, which is uniquely characterized by Step 3. By Step 4, if $K \leq \bar{K}$, then the equilibrium is a *non-absorbing equilibrium*, which is uniquely characterized by Step 1.

□

Outline of the Proof of Theorem 3.1 if $p = q = 0$ does not hold.

(1) Lemma A.5 shows that if the firm weakly prefers not to invest at state $t \geq K$, then he will strictly prefer not to invest from t on.

(2) Consider a *non-absorbing equilibrium*: $y_0 > 0$. By Lemma A.2, $y_i > 0$ for any $i \geq 0$. Show that $0 < y_i < 1$ for any $M \leq i \leq K-1$ and the firm does not invest at state K.

 (a) Prove by contradiction. Assume k as the smallest integer to satisfy $y_k = 1$, $a(k) > 0$ and $0 < y_{k-1} < 1$, where $M \leq k \leq K$.

 (b) Show that $y_i = 1$ for any $M \leq i \leq k-2$.

 (c) There is an integer N and a sequence $\{k_i\}_{i=0}^N$ such that (i) $k_0 = k-1$, $k_N \leq K-1$ and $k_i > k_{i-1}+1$; (2) For each $M \leq j \leq K-1$, $0 < y_j < 1$ if and only if $j \in \{k_i\}_{i=0}^N$.

 (d) Show that as $\Delta < \bar{\Delta}_{pq}$, then N is bounded below by an integer number $\underline{N}_{pq}$.

 (e) Show that if $N \geq \underline{N}_{pq}$, then y_{k_i} is increasing in k_i in such a way that $y_{k_N} > 1$, a contradiction.

(3) Step 3 and Lemma A.5 imply that $a_k = 0$ for $k \geq K$.

(4) Consider a *quasi-absorbing equilibrium*: Define $K - \bar{K}_{pq}$ as the largest integer k to satisfy $y_k > 0$. Let state $K - \bar{K}_{pq}$ play the same role as state 0 in the *non-absorbing equilibrium* described, then we have characterized the equilibrium behavior for $k \geq K - \bar{K}_{pq}$. For $0 < k \leq K - \bar{K}_{pq} - 1$, $a(k\Delta) = y(k\Delta) = 0$.

Lemma A.5: If the firm weakly prefers NI at state $t \geq K$, then he will strictly prefer NI from t on.

Proof. Assume by contradiction that $k \geq t+1$ is the smallest state in which the firm weakly prefers I. Therefore, the firm plays NI at state $k-1$. Therefore, $V_k - V_{k-1} \leq g_1(1,1) + \delta(qV_{k-1} + (1-q)V_{k+1}) - (g_1(1,1) + \delta(qV_{k-2} + (1-q)V_k) = \delta(V_{k+1} - V_k) + \delta q((V_k - V_{k-2}) - (V_{k+1} - V_{k-1}))$. Combined with $V_k - V_{k-2} \leq \frac{1}{\delta(1-p-q)}(g_1(0,1) - g_1(1,1)) \leq V_{k+1} - V_{k-1}$, we get $V_k - V_{k-1} \leq \delta(V_{k+1} - V_k)$.

(1) Show that the firm strictly prefers I at state $k+1$.

Assume that the firm weakly prefers NI at state $k+1$, then as the firm also weakly prefers NI at state $k-1$, $V_{k+1} - V_{k-1} = \delta(p(V_{k+2} - V_k) + (1-p)(V_k - V_{k-2})$. However, $V_{k+1} - V_{k-1} \geq \frac{1}{\delta(1-p-q)}(g_1(0,1) - g_1(1,1)) \geq (V_{k+2} - V_k)$ and $V_{k+1} - V_{k-1} \geq \frac{1}{\delta(1-p-q)}(g_1(0,1) - g_1(1,1)) \geq (V_k - V_{k-2})$, a contradiction. Therefore, the firm strictly prefers I at state $k+1$.

(2) Show that $V_{k+2} - V_{k+1} > V_{k+1} - V_k$.

By (1), we have $V_{k+1} - V_k = \delta(q(V_k - V_{k-1}) + (1-q)(V_{k+2} - V_{k+1}))$. Combined with $V_k - V_{k-1} \leq \delta(V_{k+1} - V_k)$, we have $V_{k+2} - V_{k+1} > V_{k+1} - V_k$.

(3) The firm strictly prefers I at state $k+2$.

Assume that the firm weakly prefers NI at period $k+2$, then $V_{k+3} - V_{k+1} \leq \frac{1}{\delta(1-p)}(g_1(0,1) - g_1(1,1)) \leq V_{k+2} - V_k$. Therefore, $V_{k+2} = V_{k+1} + (V_{k+2} - V_{k+1}) = g_1(0,1) + \delta V_k + \delta p(V_{k+2} - V_k) + (V_{k+2} - V_{k+1})$. By (2) and $V_{k+3} - V_{k+1} \leq V_{k+2} - V_k$, $V_{k+2} > g_1(0,1) + \delta V(k+1) + \delta p(V(k+3) - V(k+1))$. Therefore, the firm strictly prefers I at period $k+2$, a contradiction.

(4) The firm strictly prefers I from k on.

Keep using the argument of (3), the firm strictly prefers I at all state $i \geq k$. Therefore, for all $i \geq k+1$, $V_i = g_1(1,1) + \delta(qV_{i-1} + (1-q)V_{i+1}) > g_1(0,1) + \delta(pV_{i+1} + (1-p)V_{i-1})$. Since $\{V_i\}_{i \geq k}$ is a strictly increasing and bounded sequence, there is a limit V^* such that $V^* = g_1(1,1) + \delta(qV^* + (1-q)V^*)$. Therefore, $V^* = \frac{g_1(1,1)}{1-\delta}$. However, $V_{i+1} - V_{i-1} \geq \frac{1}{\delta(1-p-q)}(g_1(0,1) - g_1(1,1)$ implies that $0 = \lim_{t \to +\infty} V_{t+1} - V_{t-1} \geq \frac{1}{\delta(1-p-q)}(g_1(0,1) - g_1(1,1)$, a contradiction.

Therefore, the firm strictly prefers NI at state $t+1$. By induction, the firm strictly prefers NI from t on. □

Proof of Theorem 3.1 if $p = q = 0$ does not hold:

Proof. Firstly, we study *non-absorbing equilibria*: $y_0 > 0$. By Lemma A.1, $y_i > 0$ for any $i \geq 0$. Show that $0 < y_i < 1$ for any $M \leq i \leq K-1$ and the firm does not invest at state K. Prove by contradiction. Denote k as the smallest integer to satisfy $y_k = 1$, $a(k) > 0$ and $0 < y_{k-1} < 1$, where $M \leq k \leq K$. For simplicity of notation, we assume $g_1(0,1) - g_1(0,0) = 1$. Steps 1-7 lead to a contradiction.

Step 1: It is impossible that $0 < y_i < 1$ for all $i \leq k-1$.

Firstly, figure out the lower bound of z_{k-1}.

Define $M = 3 + \log(\delta(\epsilon - 1)/\epsilon)/\log(A_{pq}/\epsilon)$, where $\epsilon = \frac{1}{2\delta}(1 - A_{pq} + \sqrt{(1-A_{pq})^2 + 4A_{pq}\delta^2})$.

If $0 < y_i < 1$ for all $i \leq k-1$, then we can show by solving $y_{i+2} = \frac{1-A_{pq}}{\delta} y_{i+1} + A_{pq} y_i$ for $0 \leq i \leq k-3$ that $y_{k-2} \leq y_{k-1}$ by the definition of M.

By the same argument as lemma A.3(2), $1 \leq \frac{1}{\delta}(1 - A_{pq}) y_{k-1} + A_{pq} y_{k-2}$, By $y_{k-2} \leq y_{k-1}$,

$$z_{k-1} \geq \frac{1}{\frac{1}{\delta}(1 - A_{pq}) + A_{pq}}(1 + \frac{\gamma}{1-A}).$$

Next, figure out the upper bound of z_{k-1}.

Case 1: $0 < y_{k+1} < 1$.

Together with $0 < y_{k-1} < 1$, we can show that $z_{k+1} \geq \frac{1}{\delta}(1 - A_{pq})z_k + A_{pq}z_{k-1}$. Therefore, $z_{k-1} \leq \frac{1-\frac{1}{\delta}(1-A_{pq})}{A_{pq}}(1 + \frac{\gamma}{1-A})$.

Case 2: There is $i \geq 0$ such that $a(t) > 0,\ y_t = 1$ for $k+1 \leq t \leq k+i+1$. Moreover, $a_{k+i+2} = a^*(k+i+2),\ 0 < y_{k+i+2} < 1$ or $a_{k+i+2} = 0,\ y_{k+i+2} = 1$.

It is true that $V_{k+i+1} - V_{k+i-1} = \delta q(V_{k+i} - V_{k+i-2}) + \delta(1-q)(V_{k+i+2} - V_{k+i})$. Combined with $V_{k+i} - V_{k+i-2} > V_{k+i+2} - V_{k+i}$, we get $V_{k+i+1} - V_{k+i-1} - \delta(1-q)(V_{k+i+2} - V_{k+i}) < V_{k+i} - V_{k+i-2} - \delta(1-q)(V_{k+i+1} - V_{k+i-1})$.

By induction, we can show that $V_{k+i+1} - V_{k+i-1} - \delta(1-q)(V_{k+i+2} - V_{k+i}) < V_{k+1} - V_{k-1} - \delta(1-q)(V_{k+2} - V_k) = \frac{q(1-A)}{1-p-q}z_{k-1} + A(1 - z_{k-1})$. From the firm's optimality condition at state $k+i$ and $k+i+2$, $V_{k+i+2} - V_{k+i} - \delta(1-p)(V_{k+i+1} - V_{k-i+1}) \leq \delta p(V_{k+i+3} - V_{k+i+1}) < \frac{p(1-A+\gamma)}{1-p-q}$. Sum the above two inequalities and use the fact that $V_{k+i+2} - V_{k+i} > \frac{1-A+\gamma}{\delta(1-p-q)}$ and $V_{k+i+1} - V_{k+i-1} > \frac{1-A+\gamma}{\delta(1-p-q)}$, we get $z_{k-1} \leq (1 - \frac{2(1-\delta)(1-A)}{\delta(A(1-p)-q)})(1 + \frac{\gamma}{1-A})$. In all,

$$z_{k-1} \leq \max\{1 - \frac{2(1-\delta)(1-A)}{\delta(A(1-p)-q)}, \frac{1-\frac{1}{\delta}(1-A_{pq})}{A_{pq}}\}(1 + \frac{\gamma}{1-A}).$$

However, the upper bound of z_{K-1} is less than the lower bound of z_{K-1}, a contradiction.

Step 2: For any $M+1 \leq i \leq K-2$, if $y_{i-1} = 1$, then it is impossible that $0 < y_{i+1} < 1$ and $0 < y_i < 1$.

Prove by contradiction. Assume that $y_{i-1} = 1$, $0 < y_{i+1} < 1$ and $0 < y_i < 1$.

Case 1: $0 < y_{i-2} < 1$.

Prove by contradiction and assume that $0 < y_{i+1} < 1$, $0 < y_i < 1$. We can show that

$$z_{i+1} = (\frac{1}{\delta}\frac{1-A}{1-q-Ap} - \delta q)z_i + \frac{A(1-p-q)}{1-q-Ap}(1 + \frac{\gamma}{1-A}) + \frac{\delta q(A(1-p)-q)}{1-q-Ap}z_{i-2}.$$

By Assumption 3.1: $\frac{1-A}{1-q-Ap} - q > 0$, it is true that $\frac{1}{\delta}\frac{1-A}{1-q-Ap} - \delta q > 0$. Together with the fact that $z_i > z_{i-2} \geq \delta(1 + \gamma/(1-A))$, we have

$$\begin{aligned}\frac{z_{i+1}}{1+\frac{\gamma}{1-A}} &> (\frac{1}{\delta}\frac{1-A}{1-q-Ap} - \delta q)\delta + \frac{A(1-p-q)}{1-q-Ap} + \frac{\delta q(A(1-p)-q)}{1-q-Ap}\delta \\ &= \frac{(1-\delta^2 q)(1-A)}{1-q-Ap} + \frac{A(1-p-q)}{1-q-Ap} > 1,\end{aligned}$$

a contradiction.

Case 2: $y_{i-2} = 1$.

Assume that $y_t = 1$ for $j \leq t \leq i-2$ and $0 < y_{j-1} < 1$. We can show that

$$V_i - V_{i-2} - \delta q(V_{i-1} - V_{i-3}) = \frac{1-q}{1-p-q}(1-A)z_i - A(1-z_i).$$
$$V_{j+1} - V_{j-1} - \delta(1-q)(V_{j+2} - V_j) = \frac{q}{1-p-q}(1-A)z_{j-1} + A(1-z_{j-1}).$$
$$V_{i-1} - V_{i-3} - \delta(1-q)(V_i - V_{i-2}) < V_{j+1} - V_{j-1} - \delta(1-q)(V_{j+2} - V_j).$$

Sum up the above three expressions and we get $(1-\delta(1-q))(V_i - V_{i-2}) + (1-\delta q)(V_{i-1} - V_{i-3}) < \frac{1-A}{1-p-q}z_i + (\frac{q(1-A)}{1-p-q} - A)(z_{j-1} - z_i)$. By $V_i - V_{i-2} < V_{i-1} - V_{i-3}$, we have $V_i - V_{i-2} < \frac{1}{2-\delta}(\frac{1-A}{1-p-q}z_i + (\frac{q(1-A)}{1-p-q} - A)(z_{j-1} - z_i))$. By the firm's optimality condition at state i and $i+1$,

$$z_{i+1} = \frac{1}{\delta}\frac{1-A}{1-q-pA}z_i + \frac{(1-p-q)A}{1-q-Ap}z_{i-1} - \frac{\delta q(1-p-q)}{1-q-pA}(V_i - V_{i-2}).$$
$$z_{i+1} > (\frac{1}{\delta}\frac{1-A}{1-q-Ap} - \frac{\delta q}{2-\delta})z_i + \frac{A(1-p-q)}{1-q-Ap}(1 + \frac{\gamma}{1-A}) + \frac{\delta q}{2-\delta}\frac{A(1-p)-q}{1-q-Ap}z_{j-1}.$$

By Assumption 3.1: $\frac{1-A}{1-q-Ap} - q > 0$, it is true that $\frac{1}{\delta}\frac{1-A}{1-q-Ap} - \frac{\delta q}{2-\delta} > 0$. Together with $z_i > z_{j-1} \geq \delta(1 + \frac{\gamma}{1-A})$, we have $z_{i+1} > 1 + \frac{\gamma}{1-A}$, a contradiction.

Step 3: Show that $y_i = 1$ for any $M \leq i \leq k-2$.

We know $0 < y_{k-1} < 1$. Assume that $0 < y_{k-2} < 1$. By Step 1, it is impossible that $y_i < 1$ for all $i \leq k-2$, thus there exists $i \leq k-3$ such that $y_i = 1$. Let i^* be the largest one to satisfy the above condition. Then, $0 < y_{i^*+1} < 1$, $0 < y_{i^*+2} < 1$ and $y_{i^*} = 1$, a contradiction to Step 2.

In all, $y_{k-2} = 1$. By the definition of k, $y_i = 1$ for any $M \leq i \leq k-2$.

Step 4: There is an integer N and a sequence $\{k_i\}_{i=0}^N$ such that

(1) $k_0 = k-1$, $k_N \leq K-1$ and $k_i > k_{i-1} + 1$.

(2) For each $M \leq j \leq K-1$, $0 < y_j < 1$ if and only if $j \in \{k_i\}_{i=0}^N$.

Define $k_0 = k-1$. Construct an increasing sequence $\{k_i\}$ as below. For each $i \geq 0$, let k_{i+1} be the smallest $t \geq k_i + 1$ such that $0 < y_t < 1$. Then, $0 < y_{k_{i+1}} < 1$ and $y_{k_{i+1}-1} = 1$. By step 2, $y_{k_{i+1}+1} = 1$. Therefore, $k_{i+2} > k_{i+1} + 1$. Together with Step 3, we get the result.

Step 5: Show that it is impossible to have more than $\lceil \frac{\log(D)}{\log(x_1)} + 2 \rceil$ consecutive states in which $y(i) = 1$ and $a(i) > 0$, where D is defined below.

Prove by contradiction. Define $n = i_1 - i_0$. For all $i_0 \le i \le i_1$, $V_i = g_1(1,1) + \delta(qV_{i-1} + (1-q)V_{i+1})$. Define $W_i = V_i - V_{i-2}$, then for all $i_0 + 1 \le i \le i_1 - 1$, $W_{i+1} = \frac{1}{\delta(1-q)} W_i - \frac{q}{1-q} W_{i-1}$. Therefore, $W_i = \lambda_1 x_1^{i-i_0} + \lambda_2 x_2^{i-i_0}$, where $x_1 = \frac{1-\sqrt{1-4\delta^2 q(1-q)}}{2\delta(1-q)} < 1$ and $x_2 = \frac{1+\sqrt{1-4\delta^2 q(1-q)}}{2\delta(1-q)} > 1$. We can show that $W_i > W_{i+1}$, which implies that $x_1^{n-1} > \frac{(\lambda_1 x_1^{n-1} + \lambda_2 x_2^{n-1})(x_2-1)}{\lambda_1(x_2-x_1)}$.

Next, figure out the upper bound of λ_1. Assume that $0 < y_{i_0-1} < 1$, then $y_{i_0-1} \ge \delta$. Therefore, $\lambda_1 + \lambda_2 = W(i_0) = \delta(1-q)W(i_0+1) + \frac{q(1-A)}{1-p-q} y_{i_0-1} + A(1-y_{i_0-1}) \le \delta(1-q)(\lambda_1 x_1 + \lambda_2 x_2) + \frac{q(1-A)\delta}{1-p-q} + A(1-\delta)$. Because $\lambda_2 > 0$ and $\delta(1-q)x_2 - 1 < 0$, then $\lambda_1 < \frac{\frac{q(1-A)\delta}{1-p-q} + A(1-\delta)}{1-\delta(1-q)x_1}$.

By $\lambda_1 x_1^{n-1} + \lambda_2 x_2^{n-1} > \frac{1-A}{\delta(1-p-q)}$, then

$$x_1^{n-1} > \frac{(\frac{1-A}{\delta(1-p-q)})(x_2-1)}{\frac{\frac{q(1-A)\delta}{1-p-q}+A(1-\delta)}{1-\delta(1-q)x_1}(x_2-x_1)} \equiv D.$$

Therefore, $n+1 < \frac{\log(D)}{\log(x_1)} + 2$.

Step 6: There exists $\bar{\Delta}_{pq} > 0$ such that for all $\Delta < \bar{\Delta}_{pq}$, $N > \underline{N}_{pq} \equiv \lceil \frac{\log(\frac{2}{2+\delta})}{\log(A_{pq})} \rceil$.

Assume that $N \le \underline{N}_{pq} - 1$. There are $K - M - N - 1$ states in which $y_i = 1$ for $M \le i \le K-1$. Because there are $N+1$ states in which $y_i < 1$, then there exists a sequence of consecutive states in which $y_i = 1$ with the number at least $\frac{K-M-N-1}{N+2}$. By Step 5, $\frac{K-M-N-1}{N+2} \le \frac{\log(D)}{\log(x_1)} + 2$. Therefore,

$$K - M < (N+2)(\frac{\log(D)}{\log(x_1)} + 3) < (\frac{\log(\frac{2}{2+\delta})}{\log(A_{pq})} + 2)(\frac{\log(D)}{\log(x_1)} + 3).$$

a contradiction to $\bar{\Delta}_{pq} > 0$ because $\lim_{\Delta\to} LHS\Delta > 0 = \lim_{\Delta\to} RHS\Delta$.

Step 7: Show that $z_{k_{i+1}} > \frac{2-\delta}{\delta}(1-A_{pq})(1+\frac{\gamma}{1-A}) + A_{pq} z_{k_i}$ and $z_{k_0} \ge \delta(1+\frac{\gamma}{1-A})$. By $N > \underline{N}_{pq}$, $z_{k_N} > (\frac{2-\delta}{\delta} - A_{pq}^N(\frac{2-\delta}{\delta} - \delta))(1+\frac{\gamma}{1-A}) > 1 + \frac{\gamma}{1-A}$, a contradiction.

Assume that $y_t = 1$ for $k_i + 1 \le t \le k_{i+1} - 1$ and $y_{k_i}, y_{k_{i+1}} \in (0,1)$. We can show that

$$V_{k_{i+1}} - V_{k_{i+1}-2} - \delta q(V_{k_{i+1}-1} - V_{k_{i+1}-3}) = \frac{1-q}{1-p-q}(1-A)z_{k_{i+1}} - A(1-z_{k_{i+1}}).$$

$$V_{k_i} - V_{k_i-2} - \delta(1-q)(V_{k_i+1} - V_{k_i-1}) = \frac{q}{1-p-q}(1-A)z_{k_i} + A(1-z_{k_i}).$$

$$V_{k_{i+1}-1} - V_{k_{i+1}-3} - \delta(1-q)(V_{k_{i+1}} - V_{k_{i+1}-2}) < V_{k_i} - V_{k_i-2} - \delta(1-q)(V_{k_i+1} - V_{k_i-1}).$$

Sum up and we get $(1-\delta(1-q))(V_{k_{i+1}} - V_{k_{i+1}-2}) + (1-\delta q)(V_{k_{i+1}-1} - V_{k_{i+1}-3}) < \frac{1-A}{1-p-q} z_{k_{i+1}} + (\frac{q(1-A)}{1-p-q} - A)(z_{k_i} - z_{k_{i+1}})$. By $V_{k_{i+1}} - V_{k_{i+1}-2} \ge \frac{1}{\delta(1-p-q)}(1-A)$ and $V_{k_{i+1}-1} - V_{k_{i+1}-3} \ge$

$\frac{1}{\delta(1-p-q)}(1-A)$, we have $z_{k_{i+1}} > \frac{2-\delta}{\delta}(1-A_{pq})(1+\frac{\gamma}{1-A}) + A_{pq}z_{k_i}$. It is trivial that $z_{k_0} \geq \delta(1+\frac{\gamma}{1-A})$.

In all, we have shown that $0 < y_i < 1$ for any $M \leq i \leq K-1$ and the firm does not invest at state K. By Lemma A.5, the firm strictly prefers not to invest at all states $t \geq K$. By the similar argument as in Lemma A.3(1), we can show that for all $M \leq k \leq K-2$,

$$z_{k+1} = \frac{1}{\delta}(1-A_{pq})z_k + A_{pq}z_{k-1}.$$

Next, we study the *quasi-absorbing equilibrium.* By Lemma A.1, there exists $0 \leq \bar{K}_{pq} \leq K-1$ such that if $0 \leq k \leq K - \bar{K}_{pq} - 1$, then $a(k\Delta) = y(k\Delta) = 0$.

If $k \geq K - \bar{K}_{pq}$, then $y_k > 0$. Then, we treat state $K - \bar{K}_{pq}$ as state 0 in Steps 1-8 and get the same characterization as in the *non-absorbing equilibrium.*

□

Appendix B. Proofs for Section 3.2

Proof of Theorem 3.2:

Proof. **Step 1**: If $\Delta \to 0$, then it is true that $\Delta < \bar{\Delta}_{pq}$ for any $p, q \geq 0$. Therefore, for any p and q, the equilibrium is characterized as in Theorem 3.1 and Theorem 3.2. By taking $\Delta \to 0$ for the analytic solution of the equilibrium, we can show that there is a unique limiting equilibrium.

Step 2: Take the limit $\Delta \to 0$ and figure out $y(X)$ and $V(X)$, where $y(X) = \lim_{\Delta\to 0,\ k\Delta\to X} y(k\Delta)$ and $V(X) = \lim_{\Delta\to 0,\ k\Delta\to X} V(k\Delta)$. Define $z(X) = y(X) + \frac{\gamma}{1-A}$.

First, we study the *non-absorbing equilibrium* ($X^* \geq \bar{X}_{pq}$).

For any state $0 < X < X^*$, the firm is indifferent between NI and I: $V(X) = (1-\delta)g_1(0, y(X)) + \delta(pV(X+\Delta) + (1-p)V(X-\Delta)) = (1-\delta)g_1(1, y(X)) + \delta(qV(X-\Delta) + (1-q)V(X+\Delta))$. Let $\Delta \to 0$, $V'(X) = r\frac{1-A}{1-2q+(1-2p)A}(V(X) - g_1(0,0) + \frac{\gamma}{1-A}g_1(0,1))$. Therefore, $V(X) = Ce^{r\frac{1-A}{1-2q+(1-2p)A}X} + g_1(0,0) - \frac{\gamma}{1-A}g_1(0,1)$. Next, figure out the boundary condition by relating $V(X)$ to $y(X)$ and using the fact that $y(X^*) = 1$. As $(1-\delta)(g_1(0,y(X)) - g_1(1,y(X))) = \delta(1-p-q)(V(X+\Delta) - V(X-\Delta))$, $z(X) = \frac{2(1-p-q)}{r(1-A)g_1(0,1)}V'(X) = C_1e^{\frac{1-A}{1+A}rX}$. By $y(X^*) = 1$, we get

$$z(X) = e^{-\frac{1-A}{1-2q+(1-2p)A}r(X^*-X)}(1+\frac{\gamma}{1-A})\ \forall 0 < X < X^*.$$

Therefore, $V'(X) = \frac{r(1-A)g_1(0,1)}{2(1-p-q)} e^{-\frac{1-A}{1-2q+(1-2p)A} r(X^*-X)}$ for all $0 < X < X^*$. Solve C by using the above equation and $V(X) = Ce^{\frac{1-A}{1+A}rX}$, then

$$V(X) = (\frac{1-2q+(1-2p)A}{2(1-p-q)}(1+\frac{\gamma}{1-A})e^{-r\frac{1-A}{1-2q+(1-2p)A}(X^*-X)} - \frac{\gamma}{1-A})g_1(0,1) \; \forall 0 < X < X^*.$$

For $X > X^*$, $V(X) = (1-\delta)g_1(0,1) + \delta((1-p)V(X-\Delta) + pV(X+\Delta))$. Let $\Delta \to 0$, then $(1-2p)V'(X) = r(g_1(0,1) - V(X))$, thus $V(X) = g_1(0,1) - C_2 e^{-\frac{r}{1-2p}X}$. By $V(X^*-) = V(X^*+)$, we get

$$V(X) = \left(1 - \frac{(1-A)(1-2p)}{2(1-p-q)}(1+\frac{\gamma}{1-A})e^{-\frac{r}{1-2p}(X-X^*)}\right) g_1(0,1) \; \forall X > X^*.$$

Next, we study the *quasi-absorbing equilibrium* ($X^* < \bar{X}_{pq}$).

If $X^* - \bar{X}_{pq} < X < X^*$, the result is the same as in the *non-absorbing equilibrium*.

If $0 < X < X^* - \bar{X}_{pq}$, then $a(X) = y(X) = 0$. Next, figure out $V(X)$.

Consider all states $0 \le i \le I$ in the reputation absorbing state, where $I = \frac{X^* - \bar{X}_{pq}}{\Delta}$. For all $1 \le i \le I$, $V(i) = \delta(pV(i+1) + (1-p)V(i-1))$. Then, $V(i) = C_1 x_1^i + C_2 x_2^i$, where $x_1 < 1$ and $x_2 > \frac{1-p}{p}$ are two roots of $x^2 - \frac{1}{p\delta}x + \frac{1-p}{p} = 0$. If $C_2 \neq 0$, then $V(i)$ will diverge as $\Delta \to 0$, a contradiction. Therefore, $V(i) = C_1 x_1^i$.

Assume by contradiction that $V(0) \neq 0$. By $V(1) - V(0) = \delta p(V(2) - V(1))$ and $V(i) = C_1 x_1^i$, we get $V(1) = \frac{1}{\delta p}V(0)$. By $V(0) = \delta(pV(1) + (1-p)V(0))$, $V(1) = \frac{1-\delta+\delta p}{\delta p}V(0)$, a contradiction. Therefore, $V_0 = 0$ and thus $V(i) = 0$ for any $1 \le i \le I$. Therefore, in the limit, $V(X) = 0$ for any $0 < X < X^* - \bar{X}_{pq}$.

Step 3: Determine $\bar{X}_{pq}$.

We have shown that $V(X) = 0$, for any $0 < X \le X^* - \bar{X}_{pq}$. By continuity of $V(X)$ at $X^* - \bar{X}_{pq}$, $V(X^* - \bar{X}_{pq}) = 0$. Therefore,

$$\bar{X}_{pq} = \frac{1}{r}\log(\frac{1-2q+(1-2p)A}{2(1-p-q)}\frac{1-A+\gamma}{\gamma})\frac{1-2q+(1-2p)A}{1-A}.$$

□

Proof of Corollary 3.3:

Proof. The only inequality that is not trivial is that $\frac{\partial \bar{X}_{pq}}{\partial A} > 0$.

In order to show that $\partial \bar{X}_{pq}/\partial A > 0$, we only need

$$\frac{\partial \log(\frac{1-A+\gamma}{\gamma})\frac{1-2q+A(1-2p)}{1-A}}{\partial A} = \log(\frac{1-A+\gamma}{\gamma})\frac{2(1-p-q)}{(1-A)^2} - \frac{\gamma}{1-A+\gamma}\frac{1-2q+A(1-2p)}{1-A} > 0,$$

which is equivalent to $\frac{1-A+\gamma}{\gamma}\log(\frac{1-A+\gamma}{\gamma}) > \frac{(1-A)(1-2q+A(1-2p))}{2(1-p-q)}$. Since $0 < \gamma < A$ and $\frac{1-A+\gamma}{\gamma}\log(\frac{1-A+\gamma}{\gamma})$ is decreasing in γ , then we only need to show the above inequality holds if $\gamma = A$, which means that $\frac{1}{A}\log(\frac{1}{A}) > \frac{(1-A)(1-2q+(1-2p)A)}{2(1-p-q)}$. Define $f(x) = x\log(x) - \frac{(1-1/x)(1-2q+(1-2p)/x)}{2(1-p-q)}$. We need to show that $f(x) > 0$ for all $x > 1$ since $A < 1$. Since $f(1) = 0$, then $f'(x) = 1 + \log(x) + \frac{q-p}{(1-p-q)x^2} - \frac{1-2p}{(1-p-q)x^3} > 0$ implies that $f(x) > 0$ for all $x > 1$.

□

Appendix C. Proofs for Section 4.1

In this section, denote $V(k\Delta)$, $y(k\Delta)$ as V_k and y_k. For notational convenience, we use $g_1(a, y)$ instead of $(1-\delta)g_1(a, y)$.

Lemma C.1: If the firm weakly prefers I in state $k \geq K$ and $V_k \geq V_{k+1}$, then in each state $i = 0, 1, \ldots, k-1$, we have (1) the firm weakly prefers I; (2) $y_i = y_k = 1$; and (3) $V_{i-1} \geq V_i$.

Proof. We know that $y_k = 1$ for each $t \geq K$. Assume, for induction, that, for $i = k+1, \ldots, t$, the three properties hold. Consider $i = k$. Prove (2) by contradiction, assume that $y_k < y_{k+1} = 1$.

Case 1: $a(k\Delta) > a^*(k\Delta)$.

It is optimal for the buyers to choose B, so $y_k = 1$, a contradiction.

Case 2: $a(k\Delta) \leq a^*(k\Delta)$.

Then, $V_k = g_1(0, y_k) + \delta((1-p)V_0 + pV_{k+1}) \geq g_1(1, y_k) + \delta((1-q)V_{k+1} + qV_0)$. By submodularity, $g_1(0, y_k) - g_1(1, y_k) < g_1(0, 1) - g_1(1, 1)$, thus $g_1(0, 1) + \delta((1-p)V_0 + pV_{k+1}) > g_1(1, 1) + \delta((1-q)V_{k+1} + qV_0)$. Therefore, $V_{k+1} = g_1(0, 1) + \delta((1-p)V_0 + pV_{k+1}) + \delta p(V_{k+2} - V_{k+1}) > g_1(1, 1) + \delta((1-q)V_{k+1} + qV_0) + \delta p(V_{k+2} - V_{k+1}) \geq g_1(1, 1) + \delta((1-q)V_{k+2} + qV_0)$. The last inequality uses the fact that $V_{k+1} \geq V_{k+2}$. Therefore, the firm strictly prefers NI in state $k+1$, a contradiction. Therefore, we have proved (2). Then, (1) and (3) holds trivially.

□

Corollary C.1: For some $t \geq K$, if $V_t \geq V_{t+1}$, then the firm strictly prefers action NI in state $t \geq K$ and $a(t\Delta) = 0$.

Proof. If the firm weakly prefers action I in state $t \geq K$ and $V_t \geq V_{t+1}$, then by Lemma C.1, $y_i = 1$ for $i = 1, 2, \ldots, t-1$. It is obvious that $y_i = 1$ for $i \geq t$, because $i \geq t \geq K$. In all,

$y_i = 1$ for all state $i \geq 1$. Therefore, the buyer's strategy does not depend on the history of the game. As a result, the firm would strictly prefer action NI, a contradiction. □

Lemma C.2: For some $t \geq K$, if $V_t < V_{t+1}$, then the firm strictly prefers action NI in state $t \geq K$ and $a(t) = 0$.

Proof. Assume that the firm weakly prefers I at $t \geq K$ and $V_t < V_{t+1}$.

Case 1: $V_i < V_{i+1}$ for all $i \geq t$.

Then, $\{V_i\}_{i=t}^{+\infty}$ is a strictly increasing and bounded sequence and assume the limit is V^*. Furthermore, for all $i \geq t$, $V_i = g_1(1,1) + \delta((1-q)V_{i+1} + qV_0)$. Let $t \to +\infty$, then for all $i \geq t$ $V_i < V_{i+1} < V^* = \frac{g_1(1,1)+\delta V_0}{1-\delta+\delta q}$. As $V_i = g_1(1,1) + \delta((1-q)V_{i+1} + qV_0) > g_1(1,1) + \delta((1-q)V_i + qV_0)$, then $V_i > \frac{g_1(1,1)+\delta V_0}{1-\delta+\delta q}$, a contradiction.

Case 2: $V_i \geq V_{i+1}$ for some $i > t$.

Assume i^* is the smallest $i > t$ such that $V_i \geq V_{i+1}$. Therefore, $V_t < V_{t+1} < \ldots < V_{i^*}$.

If the firm weakly prefer I at i^* and $V_{i^*} \geq V_{i^*+1}$, by Lemma C.1, we know that $V_i \geq V_{i+1}$ for all $i \leq i^*$, a contradiction to $V_t < V_{t+1}$.

If the firm strictly prefer NI at i^*, then $V_{i^*} = g_1(0,1) + \delta((1-p)V_0 + pV_{i^*+1}) > g_1(1,1) + \delta((1-q)V_{i^*+1} + qV_0)$. Because the firm weakly prefer I at $t \geq K$, $V_t = g_1(1,1) + \delta((1-q)V_{t+1} + qV_0) \geq g_1(0,1) + \delta((1-p)V_0 + pV_{t+1})$. Therefore, $V_{i^*+1} < V_{t+1}$. Since $V_t < V_{i^*}$, then $V_{t+1} < V_{i^*+1}$, a contradiction.

In all, we have shown that if $V_t < V_{t+1}$, then the firm strictly prefers action NI in state $t \geq K$ and $a(t\Delta) = 0$.

□

Corollary C.2: The firm strictly prefers action NI in state $t \geq K$ and $V_t = V_K$ for all $t \geq K$.

Proof. By Corollary C.1 and Lemma C.2, the firm strictly prefers action NI in state $t \geq K$. Therefore, $V_{K+1} - V_K = (\delta p)^{n-1}(V_{K+n} - V_{K+n-1})$. As $V_{K+n} - V_{K+n-1}$ is bounded, then $V_{K+1} - V_K = \lim_{n\to+\infty}(\delta p)^{n-1}(V_{K+n} - V_{K+n-1}) = 0$. Therefore, for all $t \geq K$, $V_t = V_K = g_1(0,1) + \delta((1-p)V_0 + pV_K)$. □

Lemma C.3: If for some $j < K$, $y_{j+1} > 0$, then y_i is strictly increasing for all $j \leq i \leq K$.

Proof. By Corollary C.2, we replace V_K with V_{K+1} in the following proof. Firstly, show that $y_{K-1} < y_K$ and $V_{K-1} < V_K$.

If $y_{K-1} = 0$, then $y_{K-1} < y_K$ holds. Furthermore, $V_{K-1} = g_1(0,0) + \delta((1-p)V_0 + pV_K) < g_1(0,1) + \delta((1-p)V_0 + pV_K) = V_K$.

If $y_{K-1} > 0$, then $a((K-1)\Delta) \geq a^*((K-1)\Delta) > 0$. Then, $g_1(1, y_{K-1}) + \delta((1-q)V_K + qV_0) \geq g_1(0, y_{K-1}) + \delta((1-p)V_0 + pV_K)$. As the firm strictly prefers NI in state K, $V_K = g_1(0, y_K) + \delta((1-p)V_0 + pV_K) > g_1(1, y_K) + \delta((1-q)V_K + qV_0)$. Sum up the above two inequality, $g_1(0, y_K) - g_1(1, y_K) > g_1(0, y_{K-1}) - g_1(1, y_{K-1})$. By submodularity, $y_{K-1} < y_K$. Therefore, $V_{K-1} = g_1(1, y_{K-1}) + \delta((1-p)V_K + pV_0) \leq g_1(1, y_K) + \delta((1-q)V_K + qV_0) < V_K$.

In all, we have shown that $y_{K-1} < y_K$ and $V_{K-1} < V_K$.

Prove by contradiction. Suppose that $y_i > 0$ and $y_i \leq y_{i-1}$. Let i^* be the largest state such that $0 < y_{i^*} \leq y_{i^*-1}$. Since $y_{i^*} < y_{i^*+1}$, $y_{i^*} < 1$. Therefore, $a(i^*\Delta) = a^*(i^*\Delta)$ and $a((i^*-1)\Delta) \geq a^*((i^*-1)\Delta)$. Furthermore, $y_i > 0$ for any $i \geq i^*$ means that $a(i\Delta) \geq a^*(i\Delta)$ for any $i \geq i^*$. Therefore, for any $i \geq i^*$, we have

$$V_i = (g_1(1, y_i) + \delta qV_0) + \ldots + (\delta(1-q))^{K-i-1}(g_1(1, y_{K-2}) + \delta qV_0) + (\delta(1-q))^{K-i}V_{K-1}.$$

$$V_{i+1} = (g_1(1, y_{i+1}) + \delta qV_0) + \ldots + (\delta(1-q))^{K-i-1}(g_1(1, y_{K-1}) + \delta qV_0) + (\delta(1-q))^{K-i}V_K.$$

$V_{K-1} < V_K$ implies that $V_i < V_{i+1}$ for all $i \geq i^*$. Combined with the optimality condition at i^* and i^*-1, $g_1(0, y_{i^*-1}) - g_1(1, y_{i^*-1}) \leq \delta(1-p-q)(V_{i^*} - V_0) < \delta(1-p-q)(V_{i^*+1} - V_0) = g_1(0, y_{i^*}) - g_1(1, y_{i^*})$. By submodularity, $y_{i^*-1} < y_{i^*}$, a contradiction.

□

Lemma C.4: If $\delta > \frac{1-A+\gamma}{1-q-Ap}$, then $0 < y_i < 1$ and $a(i\Delta) = a^*(i\Delta)$ for each $i \leq K-1$ and $\{y_i\}_{i=0}^K$ is strictly increasing in i.

Proof. **Step 1:** $y_0 > 0$.

Assume, by contradiction, that $y_0 = 0$, then $a(0) \leq a^*(0) < 1$. Therefore, $V_0 = g_1(0,0) + \delta((1-p)V_0 + pV_1) \geq g_1(1,0) + \delta((1-q)V_1 + qV_0)$. Then, $V_0 \leq \frac{g_1(0,0)}{1-\delta} + \frac{p}{1-p-q}\frac{g_1(0,0)-g_1(1,0)}{1-\delta}$. Because $\delta > \frac{1-A+\gamma}{1-q-Ap}$ and $y_K = 1$, we can show that

$$g_1(0,1) + \delta((1-p)V_0 + p\frac{g_1(1,1) + \delta V_0}{1-\delta(1-q)}) < \frac{g_1(1,1) + \delta V_0}{1-\delta(1-q)}.$$

Using the fact that $V_K > \frac{g_1(1,1)+\delta V_0}{1-\delta(1-q)}$, the above inequality implies that $g_1(0,1)+\delta((1-p)V_0+pV_K) < V_K$, a contradiction to the fact that the firm strictly prefers NI in state K.

Step 2: $y_1 > 0$.

Next, assume, by contradiction, that $y_1 = 0$. Then, $a(\Delta) \leq a^*(\Delta) < 1$, so NI is an optimal choice for the firm in state Δ. Therefore, $V_1 = g_1(0,0) + \delta((1-p)V_0 + pV_2) \geq g_1(1,0) + \delta((1-q)V_2 + qV_0)$. Then, $V_2 - V_0 \leq \frac{g_1(0,0)-g_1(1,0)}{(1-p-q)\delta}$.

$y_0 > 0$ implies $a(0) \geq a^*(0) > 0$, so I is an optimal choice for the firm in state 0. Therefore, $V_1 - V_0 \geq \frac{g_1(0,y_0)-g_1(1,y_0)}{(1-p-q)\delta} \geq \frac{g_1(0,0)-g_1(1,0)}{(1-p-q)\delta} \geq V_2 - V_0$. Then, $V_2 \leq V_1$. Therefore,

$$\begin{aligned}
V_0 &= g_1(1,y_0) + \delta((1-q)V_1 + qV_0) \leq g_1(1,y_0) + \delta V_1 \\
&= g_1(1,y_0) + \delta g_1(0,0) + \delta^2((1-p)V_0 + pV_2) \leq g_1(1,y_0) + \delta g_1(0,0) + \delta^2((1-p)V_0 + pV_1) \\
&< g_1(0,y_0) + \delta g_1(0,y_0) + \delta^2((1-p)V_0 + pV_1) < g_1(0,y_0) + \delta V_0 \leq V_0,
\end{aligned}$$

a contradiction.

By Lemma C.3, $y_1 > 0$ implies that $\{y_i\}_{i=0}^K$ is strictly increasing in i. Therefore, $y_i > 0$ for each $i < K$. Because $y_K = 1$ and $\{y_i\}_{i=0}^K$ is strictly increasing in i, $y_i < 1$ for each $i < K$. Therefore, $0 < y_i < 1$ for each $i < K$ implies that $a(i\Delta) = a^*(i\Delta)$ for each $i < K$.

□

Proof of Theorem 4.1:

Proof. It is obvious that $y(t) = 1$ for each $t \geq K$. By Corollary C.1, the firm strictly prefers action NI in state $t \geq K$. Then, we have proved (2). Lemma C.4 proved (1). Then, let's characterize y_k for $0 \leq k \leq K-1$. The optimality condition at state $k\Delta$ is $V_k = g_1(0,y_k) + \delta((1-p)V_0 + pV_{k+1}) = g_1(1,y_k) + \delta((1-q)V_{k+1} + qV_0)$. Therefore, $V_{k+1} - V_0 = \frac{g_1(0,y_k)-g_1(1,y_k)}{\delta(1-p-q)}$ and $V_k - V_0 = g_1(0,y_k) - (1-\delta)V_0 + \delta p(V_{k+1} - V_0)$. Together with $(1-\delta)V_0 = g_1(0,y_0) + \frac{p}{1-p-q}(g_1(0,y_0) - g_1(1,y_0))$, the above two equations imply that $y_k = \eta_1 y_{k-1} + y_0 + \eta_2$, where $\eta_1 = \frac{1-A}{\delta(1-q-pA)}$, $\eta_2 = \frac{\gamma}{\delta(1-q-pA)}$. Then, we can solve for y_k for $0 \leq k \leq K-1$ by $y_K = 1$. □

Proof of Proposition 4.2:

Proof. Because $\lim_{k\to+\infty} y(k) = 1$, then $y_k = \eta_1 y_{k-1} + y_0 + \eta_2$ implies that $y_0 = 1 - \eta_1 - \eta_2 = 1 - \frac{1-A+\gamma}{\delta(1-q-pA)}$. Therefore, for any $k \geq 0$, $y_k = 1 - (\eta_1 + \eta_2)\eta_1^k$.

Since $\frac{\partial \eta_1}{\partial q} > 0$, $\frac{\partial \eta_1}{\partial p} > 0$, $\frac{\partial \eta_2}{\partial q} > 0$, $\frac{\partial \eta_2}{\partial p} > 0$, then $\frac{\partial y(k)}{\partial q} < 0$, $\frac{\partial y(k)}{\partial p} < 0$. Since $\frac{\partial \eta_1}{\partial A} < 0$, $\frac{\partial \eta_2}{\partial \gamma} > 0$, then $\frac{\partial y(k)}{\partial A} > 0$, $\frac{\partial y(k)}{\partial \gamma} < 0$. Since $\frac{\partial \eta_1}{\partial \delta} < 0$, $\frac{\partial \eta_2}{\partial \delta} < 0$, then $\frac{\partial y(k)}{\partial \delta} < 0$, $\frac{\partial y(k)}{\partial \delta} < 0$. □

Proof of Proposition 4.3:

Proof. Assume that $y(0) > 0$, then we have shown that in the limit case, $y_0 = 1 - \frac{1-A+\gamma}{\delta(1-q-pA)}$. If Assumption 4.2 is violated, then $y_0 \leq 0$, a contradiction. Therefore, $y_0 = 0$ in the limit case. Therefore, $V_1 - V_0 \leq \frac{1}{\delta(1-p-q)}(g_1(0,0) - g_1(1,0))$. Since $y_{k^*} > 0$, by Lemma C.3, we know that $0 < y_k < 1$ for all $k^* \leq k \leq K-1$. Therefore, $V_{k+1} - V_0 = \frac{g_1(0,y_k) - g_1(1,y_k)}{\delta(1-p-q)}$ and $V_{k+1} - V_0 = g_1(0, y_{k+1}) - (1-\delta)V_0 + \delta p(V_{k+2} - V_0)$. Together with $(1-\delta)V_0 = g_1(0,0) + \delta p(V_1 - V_0)$, the above two equations imply that for all $k^* \leq k \leq K-1$,

$$y_{k+1} = \frac{1-A}{\delta(1-q-pA)} y_k + \frac{(1/\delta - p)\gamma}{1-q-pA} + \frac{\delta p(1-p-q)}{1-q-pA} \frac{V_1 - V_0}{g_1(0,1) - g_1(1,1)}.$$

Prove by contradiction that $K - k^* \to +\infty$ as $K \to +\infty$, then $\frac{1-A}{\delta(1-q-pA)} < 1$ and $\lim_{K-k^* \to +\infty} y_{k^*+i} = 1$, which implies that

$$1 = \frac{1-A}{\delta(1-q-pA)} + \frac{(1/\delta - p)\gamma}{1-q-pA} + \frac{\delta p(1-p-q)}{1-q-pA} \frac{V_1 - V_0}{g_1(0,1) - g_1(1,1)}.$$

Together with $V_1 - V_0 \leq \frac{1}{\delta(1-p-q)}(g_1(0,0) - g_1(1,0))$, we get $\frac{1-A+\gamma}{(1-q-pA)} \geq \delta$, a contradiction to $\frac{1-A}{\delta(1-q-pA)} < 1$.

□

Appendix D. Proofs for Section 4.2

In this section, for notational convenience, we use $g_1(a,y)$ instead of $(1-\delta)g_1(a,y)$.

Lemma D.1 : Show that the buyers strictly prefer NI at all $t \geq K+1$.

Proof. Show that there are no two consecutive states $t, t+1 \geq K$ such that the firm weakly prefers I. Prove by contradiction and assume that $V_t = g_1(1,1) + \delta V_{t+1} \geq g_1(0,1) + \delta V_{t-1}$ and $V_{t+1} = g_1(1,1) + \delta V_{t+2} \geq g_1(0,1) + \delta V_t$. Then, $V_{t+2} - V_{t+1} = \frac{1}{\delta}(V_{t+1} - V_t) > \delta(V_{t+1} - V_t)$. Therefore, $V_{t+2} = V_{t+1} + (V_{t+2} - V_{t+1}) > g_1(0,1) + \delta V_t + \delta(V_{t+1} - V_t) = g_1(0,1) + \delta V_{t+1}$. Then, $V_{t+2} = g_1(1,1) + \delta V_{t+3} > g_1(0,1) + \delta V_{t+1}$. By induction, for any $i \geq t$, $V_i = g_1(1,1) + \delta V_{i+1} \geq g_1(0,1) + \delta V_{i-1}$. a contradiction.

If the buyer weakly prefers I at some $t \geq K+1$, then $V_{t+1} = g_1(1,1) + \delta V_{t+2} \geq g_1(0,1) + \delta V_t$. By the argument in the last paragraph, $V_t = g_1(0,1) + \delta V_{t-1)} > g_1(1,1) + \delta V_{t+1}$ and

$V_{t+2} = g_1(0,1) + \delta V_{t+1} > g_1(1,1) + \delta V_{t+3}$. Therefore, $\frac{1}{\delta}(g_1(0,1) - g_1(1,1)) < V_{t+2} - V_t = \delta(V_{t+1} - V_{t-1}) < g_1(0,1) - g_1(1,1)$, a contradiction. □

Lemma D.2: If $y_k = 0$, then $y_i = 0$ for any $0 \leq i \leq k-1$.

Proof. If $y_k = 0$, then $V_k = 0$. Because $V_{k-2} \geq 0$, $g_1(1,1) + \delta V_k < g_1(0,1) + \delta V_{k-2}$. Therefore, the firm does not invest in state $k-1$, then $y_{k-1} = 0$. By induction, $y_i = 0$ for any $0 \leq i \leq k-1$.

□

Lemma D.3: For some $0 \leq k \leq K-1$, $y_k = 1$ implies that $y_{k+2i} = 1$ and $y_{k+2i+1} \in (0,1)$, where $i \geq 0$, $k + 2i \leq K-1$ and $k + 2i + 1 \leq K-1$. Furthermore, if $k + 2i = K$, then $V_K = g_1(1,1) + \delta V_{K+1} > g_1(0,1) + \delta V_{K-1}$. If $k + 2i + 1 = K$, then $V_K = g_1(0,1) + \delta V_{K-1} > g_1(1,1) + \delta V_{K+1}$.

Proof. By Lemma D.2, $y_i > 0$ for $i \geq k$. Therefore, $\delta(V_{i+1} - V_{i-1}) \geq g_1(0,1) - g_1(1,1)$ for all $k \leq i \leq K$. $y_k = 1$ implies that $V_k = g_1(1,1) + \delta V_{k+1} > g_1(0,1) + \delta V_{k-1}$.

Assume by contradiction that $y_{k+2} \in (0,1)$, then $V_{k+2} = \delta(1 - y_{k+2})V_{k+2} + y_{k+2}(g_1(1,1) + \delta V_{k+3}) < g_1(1,1) + \delta V_{k+3}$. Therefore, $V_{k+2} - V_k < \delta(V_{k+3} - V_{k+1}) = g_1(0,1) - g_1(1,1)$, a contradiction to $y_{k+1} > 0$. In all, $y_{k+2} = 1$. By induction, $y_{k+2i} = 1$, where $i \geq 0$, $k + 2i \leq K-1$.

If $k + 2i = K$, assume by contradiction that $V_K = g_1(0,1) + \delta V_{K-1}$. Then, by the fact that $V_{K-2} = g_1(1,1) + \delta V_{K-1}$, $V_K - V_{K-2} = g_1(0,1) - g_1(1,1)$, a contradiction to $y_{K-1} > 0$. Thus, $V_K = g_1(1,1) + \delta V_{K+1} > g_1(0,1) + \delta V_{K-1}$.

Assume by contradiction that $y_{k+1} = 1$, then by the same argument as the last paragraph, we get $y_{k+3} = 1$. By induction, $y_{k+2i+1} = 1$, where $i \geq 0$, $k + 2i + 1 \leq K-1$. In all, $y_i = 1$ for all $k \leq i \leq K-1$. Next, show that $V_K = g_1(1,1) + \delta V_{K+1} > g_1(0,1) + \delta V_{K-1}$. Otherwise, $V_K = g_1(0,1) + \delta V_{K-1}$. Combined with $V_{K-2} = g_1(1,1) + \delta V_{K-1}$, we get $V_K - V_{K-2} = g_1(0,1) - g_1(1,1)$, a contradiction with $y_{K-1} = 1$. By Lemma D.1, $V_{K+1} = g_1(0,1) + \delta V_K$. Combined with $V_{K-1} = g_1(1,1) + \delta V_K$, we get $V_{K+1} - V_{K-1} = g_1(0,1) - g_1(1,1)$, a contradiction with $V_K = g_1(1,1) + \delta V_{K+1} > g_1(0,1) + \delta V_{K-1}$. Therefore, $y_{k+1} \in (0,1)$. By induction, $y_{k+2i+1} \in (0,1)$, where $i \geq 0$, $k + 2i + 1 \leq K-1$.

If $k + 2i + 1 = K$, assume by contradiction that $V_K = g_1(1,1) + \delta V_{K+1} \geq g_1(0,1) + \delta V_{K-1}$. By the fact that $V_{K+1} = g_1(0,1) + \delta V_K$ and $V_{K-1} = g_1(1,1) + \delta V_k$, $V_{K+1} - V_{K-1} =$

$g_1(0,1) - g_1(1,1)$, a contradiction to $V_K = g_1(1,1) + \delta V_{K+1} \geq g_1(0,1) + \delta V_{K-1}$. Thus, $V_K = g_1(0,1) + \delta V_{K-1} > g_1(1,1) + \delta V_{K+1}$. □

Lemma D.4: If $K \geq \hat{K}$, then there is a unique quasi-absorbing equilibrium. Furthermore, the necessary condition for the existence of *quasi-absorbing equilibrium* is $K \geq \hat{K}$, where $\hat{K} \equiv \lfloor \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{\delta}{1+\delta} \rfloor + 1$.

Proof. Assume $y_k = 0$ and $y_{k+1} > 0$, then by Lemma D.2, $y_i = 0$ for all $0 \leq i \leq k$ and $y_i > 0$ for all $i \geq k+1$.

Step 1: Show that in any *quasi-absorbing equilibrium* $y_i \in (0,1)$ for $1 \leq i \leq K-1$. Furthermore, the firm does not invest in state K.

Prove by contradiction, there exists $k+1 \leq m \leq K-1$ such that $y_m = 1$. Assume that m is the smallest integer that $y_m = 1$. Therefore, (1) $y_i = 0$ for all $0 \leq i \leq k$; (2) $0 < y_i < 1$ for all $k+1 \leq i \leq m-1$; (3) $y_m = 1$. By Lemma D.3, we also have (4) $y_{m+2i} = 1$ and $y_{m+2i+1} \in (0,1)$, where $m+2i, m+2i+1 < K-1$; (5) If $m+2i = K$, then $V_K = g_1(1,1) + \delta V_{K+1} > g_1(0,1) + \delta V_{K-1}$. If $m+2i+1 = K$, then $V_K = g_1(0,1) + \delta V_{K-1} > g_1(1,1) + \delta V_{K+1}$.

Case 1: $K - m$ is even.

By (5), $V_K = g_1(1,1) + \delta V_{K+1} \geq g_1(0,1) + \delta V_{K-1}$. By Lemma D.1, $V_{K+1} = g_1(0,1) + \delta V_K$. Therefore, $V_{K+1} = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2}$ and $V_K = \frac{g_1(1,1)+\delta g_1(0,1)}{1-\delta^2}$. Assume WLOG that $g_1(0,1) = 1$ and $g_1(1,1) = A$.

(1) $K - k$ is even. By (2) + (4), $V_{K-2i} - V_{K-2i-2} = \frac{1-A}{\delta}$ for $0 \leq i \leq \frac{K-k}{2} - 1$. Therefore, by $V_k = 0$, we have $V_K = \frac{(K-k)(1-A)}{2\delta}$. Then, $K - k = \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{\delta}{1+\delta}$, which is not generically true since $K - k$ is an integer number.

(2) $K - k$ is odd. It is trivial to show that $\frac{1-A}{\delta} < V_{m+1} - V_{m-1} < \frac{1-A}{\delta^2}$. By (2) and (4) $V_{m-1} = \frac{(m-k-1)(1-A)}{2\delta}$ and $V_{m+1} = V_{K+1} - \frac{(K-m)(1-A)}{2\delta^2}$. Therefore, $V_{m+1} - V_{m-1} = V_{K+1} - \frac{(K-m)(1-A)}{2\delta^2} - \frac{(m-k-1)(1-A)}{2\delta}$. In all,

$$\frac{2\delta}{1-A}V_{K+1} - \frac{(K-m)(1-\delta)}{\delta} - \frac{2(1-\delta)}{\delta} < K - k + 1 < \frac{2\delta}{1-A}V_{K+1} - \frac{(K-m)(1-\delta)}{\delta},$$

which is not generically true for δ close to 1.

We have shown that $y_i \in (0,1)$ for $1 \leq i \leq K-1$. If the firm weakly prefers to invest in state K, then $V_K = g_1(1,1) + \delta V_{K+1} \geq g_1(0,1) + \delta V_{K-1}$. By the same argument as above and let $m = K$, we reach a contradiction. In all, the firm does not invest in state K.

Case 2: $K-m$ is odd.

Then, by (5), $V_K = g_1(0,1)+\delta V_{K-1} > g_1(1,1)+\delta V_{K+1}$. We also have $V_{K-1} = g_1(1,1)+\delta V_K$. Let K play the same role as $K+1$ in Case 1, we reach a contradiction.

Step 2: If $0 < y_i < 1$ for $1 \le i \le K-1$, then $K-k = \hat{K}$, where $\hat{K} \equiv \lfloor \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{\delta}{1+\delta} \rfloor + 1$.

We need to solve the following linear system: $y_k = V_k = 0$,

$$V_i = (1-y_i)\delta V_i + y_i(g_1(1,1)+\delta V_{i+1}) = (1-y_i)\delta V_i + y_i(g_1(0,1)+\delta V_{i-1})\ \forall k+1 \le i \le K-1.$$

$$V_i = g_1(0,1) + \delta V_{i-1} > g_1(1,1) + \delta V_{i+1}\ \forall i \ge K.$$

Solve the above equations: for $2i+1, 2i+2 \le K$,

$$y_{k+2i+1} = y_{k+1} + \frac{c_1}{\delta} i,\ y_{k+2i} = \frac{c_2}{\delta} i.$$

where $c_1 \equiv (1-\delta+\delta y_1)(1-A)$ and $c_2 \equiv \frac{(1-\delta+\delta y_1)(1-A)}{A+\frac{(1+A)\delta y_1}{1-\delta}}$. The boundary condition is $y_K = 1$. Furthermore, we need $g_1(0,1)+\delta V_k > g_1(1,1)+\delta V_{k+1}$, which implies that $y_{k+1} < \frac{1-\delta}{\delta}\frac{1-A}{A}$.

Case 1: K is an odd number.

Then we can solve y_{k+1} by backward induction. $1 = y_K = y_{k+1} + \frac{c_1}{\delta}\frac{K-k-1}{2}$ implies that

(D.1)
$$y_{k+1} = \frac{\frac{2\delta}{1-A} - (1-\delta)(K-k-1)}{\delta(K-k+\frac{2}{1-A}-1)}.$$

$y_{K-1} \le 1$ implies that $\frac{c_2}{\delta}\frac{K-k-1}{2} \le 1$. Therefore, $K-k \le \frac{1+A}{1-A}\frac{\delta}{1-\delta} + \frac{1}{1+\delta}$. The optimality condition at state K: $\delta(V(K+1)-V(K-1)) < g_1(0,1)-g_1(1,1)$ implies that $K-k > \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{1}{1+\delta}$. In all,

$$K-k = \lfloor \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{1}{1+\delta} \rfloor + 1.$$

Case 2: K is an even number.

Then, we can solve y_{k+1} by backward induction.

(D.2)
$$y_{k+1} = \frac{(1-\delta)((K-k)(1-\delta) - \frac{2A}{1-A}\delta)}{\frac{2(1+A)}{1-A}\delta^2 - (K-k)\delta(1-\delta)}.$$

$y_{K-1} \le 1$ implies that $y_{k+1} + \frac{c_1}{\delta}\frac{K-k-2}{2} \le 1$. Therefore, $K-k \le \frac{1+A}{1-A}\frac{\delta}{1-\delta} + \frac{\delta}{1+\delta}$. The optimality condition at state K: $\delta(V_{K+1}-V_{K-1}) < g_1(0,1)-g_1(1,1)$ implies $K-k > \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{\delta}{1+\delta}$. In all,

$$K-k = \lfloor \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{\delta}{1+\delta} \rfloor + 1.$$

Define $\hat{K}$ as follows for δ close to 1:

$$\hat{K} \equiv \lfloor \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{\delta}{1+\delta} \rfloor + 1 = \lfloor \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{1}{1+\delta} \rfloor + 1.$$

Step 3: A necessary condition for the existence of a *quasi-absorbing equilibrium* is $K \geq \hat{K}$.

By Step 2, $K - k = \hat{K}$, then $k \geq 0$ implies that $K \geq \hat{K}$.

□

Lemma D.5: There is a unique *non-absorbing equilibrium* and the necessary condition for the existence of *non-absorbing equilibrium*is $K \leq \hat{K} - 1$.

Firstly, consider the case that K is even.

Step 1: $V_K = g_1(0,1) + \delta V_{K-1} > g_1(1,1) + \delta V_{K+1}$.

Prove by contradiction, then $0 < y_{K-1} < 1$ and $V_{K+1} = g_1(0,1) + \delta V_K > g_1(1,1) + \delta V_{K+1}$ by Lemma D.1. Show by induction that $y_{K-2i} = 1$ and $y_{K-2i-1} \in (0,1)$ for all $0 \leq i \leq K/2$. Assume that it is true for $0 \leq i \leq k$. We need to show that $y_{K-2k-2} = 1$ and $y_{K-2k-3} \in (0,1)$. $y_{K-2k-2} \in (0,1)$ implies $0 < y_i < 1$ for all $0 \leq i \leq K-2k-2$ and thus $\frac{V_{K-2k-2}}{y_{K-2k-2}} = \frac{V_0}{y_0} = \frac{g_1(0,1)}{1-\delta}$. By $y_{K-2k} = 1$ and $V_{K-2k} = g_1(1,1) + \delta V_{K-2k+1}$, we can show that $V_{K-2k} > \frac{V_{K-2k-2}}{y_{K-2k-2}} = \frac{g_1(0,1)}{1-\delta}$, a contradiction to $g_1(0,1)$ is the firm's highest stage-game payoff. Therefore, $y_{K-2k-2} = 1$. Lemma D.3 tells us that $y_{K-2k-1} \in (0,1)$ implies $y_{K-2k-3} \in (0,1)$. In all, we show that $y_{K-2i} = 1$, which implies that $y_0 = 1$ and $V_0 = \frac{g_1(0,1)}{1-\delta}$. This is impossible because $V_1 > V_0$ will be higher than the highest possible continuation payoff for $i \geq 1$.

Step 2: Figure out the equilibrium if $K \leq 2\lfloor \frac{\delta^2}{1-\delta^2} \rfloor$.

By Step 1 and Lemma D.3, $y_{K-2} \in (0,1)$. Show that $y_{K-1} = 1$. If $0 < y_{K-1} < 1$, then $0 < y_{k-2} < 1$ for all $k \leq K$ and thus $V_K = \frac{V_0}{y_0} = \frac{g_1(0,1)}{1-\delta}$, then $V_{K-1} = V_K = \frac{g_1(0,1)}{1-\delta}$, a contradiction to the fact that $V_{K-1} \leq g_1(1,1) + \delta V_K$. In all, $y_{K-1} = 1$. Therefore, $V_{K-1} = g_1(1,1) + \delta V_K$ and $V_K = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2}$. Furthermore, $\frac{V_{K-2}}{y_{K-2}} = V_K + \frac{g_1(0,1)-g_1(1,1)}{\delta} = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{g_1(0,1)-g_1(1,1)}{\delta}$.

Show that if $K \leq 2\lfloor \frac{\delta^2}{1-\delta^2} \rfloor$, then $y_{K-2i} \in (0,1)$ and $y_{K-2i-1} = 1$ for $0 \leq i \leq K/2$. Furthermore, $\frac{V_{K-2i}}{y_{K-2i}} = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{i(g_1(0,1)-g_1(1,1))}{\delta}$. By induction, this is true for $0 \leq i \leq k$. We need to show that $y_{K-2k-2} \in (0,1)$ and $y_{K-2k-3} = 1$. By Lemma D.3, $y_{K-2k} \in (0,1)$ implies that $y_{K-2k-2} \in (0,1)$. Next, show that $y_{K-2k-3} = 1$. If $0 < y_{K-2k-3} < 1$, then $0 < y_i < 1$ for all $i \leq K - 2k - 3$. Then, $\frac{V_{K-2k}}{y_{K-2k}} = \frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\delta}$. Because $K \leq 2[\frac{\delta^2}{1-\delta^2}]$, then for $k \leq \frac{K}{2}$, $\frac{V_{K-2k}}{y_{K-2k}} = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{k(g_1(0,1)-g_1(1,1))}{\delta} < \frac{g_1(0,1)}{1-\delta}$, a contradiction. Therefore, $y_{K-2k-3} = 1$ and $\frac{V_{K-2k-2}}{y_{K-2k-2}} = \frac{V_{K-2k}}{y_{K-2k}} + \frac{g_1(0,1)-g_1(1,1)}{\delta} = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{(k+1)(g_1(0,1)-g_1(1,1))}{\delta}$.

Because $V_{K-2k+2} = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} - \frac{(k-1)(g_1(0,1)-g_1(1,1))}{\delta^2} = (\frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{i(g_1(0,1)-g_1(1,1))}{\delta})y_{K-2k+2}$, for any $1 \le k \le \frac{K}{2}$,

$$y_{K-2k+2} = \frac{\delta^2(1+A\delta) - (1-\delta^2)(k-1)(1-A)}{\delta^2(1+A\delta) + (1-\delta^2)\delta(k-1)(1-A)}. \tag{D.3}$$

Next, figure out y_0. Because $\frac{V_0}{y_0} = \frac{g_1(0,1)}{1-\delta}$ and $(1-\delta)(\frac{V_2}{y_2} - \frac{V_0}{y_0}) = (g_1(0,1)-g_1(1,1)) - \delta(V_2 - V_0)$,

$$y_0 = (\frac{1+A\delta}{1+\delta} - \frac{(1-A)(K-2)(1-\delta)}{2\delta^2}) + \frac{1-\delta}{\delta}(\frac{(1-A)(K-2)(1-\delta)}{2\delta} - \frac{(1-A)(1+2\delta)}{1+\delta}).$$

Step 3: Figure out the equilibrium if $K \ge 2\lfloor \frac{\delta^2}{1-\delta^2} \rfloor + 2$.

Therefore, $\frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{K(g_1(0,1)-g_1(1,1))}{2\delta} > \frac{g_1(0,1)}{1-\delta}$. Denote $k^* < \frac{K}{2}$ as the largest integer k such that $\frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{k(g_1(0,1)-g_1(1,1))}{\delta} < \frac{g_1(0,1)}{1-\delta}$. Then, $k^* = \lfloor \frac{\delta^2}{1-\delta^2} \rfloor$. By the same argument as in Step 2, for any $1 \le k \le k^*$,

$$y_{K-2k+2} = \frac{\delta^2(1+A\delta) - (1-\delta^2)(k-1)(1-A)}{\delta^2(1+A\delta) + (1-\delta^2)\delta(k-1)(1-A)},\ y_{K-2k+1} = 1. \tag{D.4}$$

Denote $\tilde{K} = K - 2k^* - 2 = K - 2[\frac{\delta^2}{1-\delta^2}] - 2$.

(1) Show that $y_{\tilde{K}+1} = 1$. Assume by contradiction that $0 < y_{\tilde{K}+1} < 1$, then $0 < y_i < 1$ and $\frac{V_{i-1}}{y_{i-1}} = \frac{V_{i+1}}{y_{i+1}}$ for all $0 \le i \le \tilde{K}+1$. Specifically, $\frac{V_{\tilde{K}+2}}{y_{\tilde{K}+2}} = \frac{V_0}{y_0} = \frac{g_1(0,1)}{1-\delta}$. Because $V_{\tilde{K}+1} < g_1(1,1) + \delta V_{\tilde{K}+2}$ and $V_{\tilde{K}+3} = g_1(1,1) + \delta V_{\tilde{K}+4}$, then $\delta(V_{\tilde{K}+4} - V_{\tilde{K}+2}) < V_{\tilde{K}+3} - V_{\tilde{K}+1} = \frac{g_1(0,1)-g_1(1,1)}{\delta}$. Furthermore, $(1-\delta)(\frac{V_{\tilde{K}+4}}{y_{\tilde{K}+4}} - \frac{V_{\tilde{K}+2}}{y_{\tilde{K}+2}}) = g_1(0,1) - g_1(1,1) - \delta(V_{\tilde{K}+4} - V_{\tilde{K}+2})$. Therefore, $\frac{g_1(0,1)}{1-\delta} = \frac{V_{\tilde{K}+2}}{y_{\tilde{K}+2}} < \frac{V_{\tilde{K}+4}}{y_{\tilde{K}+4}} + \frac{g_1(0,1)-g_1(1,1)}{\delta} = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{k^*(g_1(0,1)-g_1(1,1))}{\delta}$, a contradiction to the definition of k^*.

(2) Show that $0 < y_i < 1$ for all $i \le \tilde{K} - 1$.

As $y_{\tilde{K}+1} = 1$, $0 < y_{\tilde{K}} < 1$. If we assume $y_{\tilde{K}-1} = 1$, then $\frac{V_{\tilde{K}}}{y_{\tilde{K}}} = \frac{g_1(0,1)+\delta g_1(1,1)}{1-\delta^2} + \frac{(k^*+1)(g_1(0,1)-g_1(1,1))}{\delta} > \frac{g_1(0,1)}{1-\delta}$, a contradiction. Therefore, $0 < y_{\tilde{K}-1} < 1$ and $0 < y_{\tilde{K}} < 1$. This implies that $0 < y_i < 1$ for all $0 \le i \le \tilde{K} - 1$.

(3) Solve for $\{y_i\}_{i=0}^{\tilde{K}-1}$.

It is trivial that $\frac{V_0}{y_0} = \frac{g_1(0,1)}{1-\delta}$. As $y_0, y_1 \in (0,1)$, then $V_1 - V_0 = V_2 - V_0 = \frac{1}{\delta}(g_1(0,1) - g_1(1,1))$, thus $V_2 = V_1$. Furthermore, $V_1 = \delta(1-y_1)V_1 + y_1(g_1(1,1) + \delta V_2)$, then $\frac{V_1}{y_1} = \frac{g_1(1,1)}{1-\delta}$. In all, for any $0 \le i \le \tilde{K}$, $\frac{V_{2i}}{y_{2i}} = \frac{g_1(0,1)}{1-\delta}$ and $\frac{V_{2i+1}}{y_{2t+1}} = \frac{g_1(1,1)}{1-\delta}$. Therefore, for any $0 \le i \le \tilde{K}$,

$$y_{2i} = y_{\tilde{K}} + \frac{(1-\delta)(1-A)}{2\delta}(\tilde{K} - 2i),\ y_{2i+1} = y_{\tilde{K}-1} + \frac{(1-\delta)(1-A)}{2\delta A}(\tilde{K} - 2i - 2). \tag{D.5}$$

Figure out $y_{\tilde{K}}$. We know that $\frac{V_{\tilde{K}}}{y_{\tilde{K}}} = \frac{g_1(0,1)}{1-\delta}$ and $(1-\delta)(\frac{V_{\tilde{K}+2}}{y_{\tilde{K}+2}} - \frac{V_{\tilde{K}}}{y_{\tilde{K}}}) = g_1(0,1) - g_1(1,1) - \delta(V_{\tilde{K}+2} - V_{\tilde{K}})$. Furthermore, we know $V_{\tilde{K}+2}$ and $y_{\tilde{K}+2}$, then

$$y_{\tilde{K}} = (\frac{1+A\delta}{1+\delta} - \frac{(1-A)(K-\tilde{K}-2)(1-\delta)}{2\delta^2}) + \frac{1-\delta}{\delta}(\frac{(1-A)(K-\tilde{K}-2)(1-\delta)}{2\delta} - \frac{(1-A)(1+2\delta)}{1+\delta}).$$

Step 4: Show that $K \leq \hat{K}+1$.

Because $k^* < \frac{\delta^2}{1-\delta^2}$ and $K - \tilde{K} - 2 = 2k^*$, then

$$y_{\tilde{K}} \leq (\frac{1+A\delta}{1+\delta} - \frac{(1-A)(K-\tilde{K}-2)(1-\delta)}{2\delta^2}) - \frac{(1-\delta)(1-A)}{\delta}.$$

Furthermore, $0 \leq y_0 = y_{\tilde{K}} - \frac{(1-\delta)(1-A)}{\delta}\frac{\tilde{K}}{2}$ implies that $y_{\tilde{K}} \geq \frac{(1-\delta)(1-A)}{\delta}\frac{\tilde{K}}{2}$. We can show that $K < \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{1-\delta}{1+\delta} = \hat{K}$, thus $K \leq \hat{K} - 1$.

Step 5: If K is odd, then denote $K^* = K+1$. It can be show that K^* plays the same role as K in previous steps in which K is even. In all, all the results in the previous steps hold for K^*, if we denote $K^* = K+1$ if K is odd and $K^* = K$ if K is even.

Proof of Theorem 4.4:

Proof. If $K \geq \hat{K}$, then by Lemma D.5, the equilibrium is a *quasi-absorbing equilibrium.* By lemma D.4, there is a unique quasi-absorbing equilibrium and the limiting equilibrium is also characterized. In all, there is a unique stationary Markov equilibrium and it is a *quasi-absorbing equilibrium.*

If $K \leq \hat{K} - 1$, then by Lemma D.4, the equilibrium is a *non-absorbing equilibrium.* By Lemma D.5, there is a unique *non-absorbing equilibrium* and the limiting equilibrium is also characterized. In all, there is a unique stationary Markov equilibrium and it is a *non-absorbing equilibrium.*

□

Proof of Proposition 4.5:

Proof. Define $\hat{X} \equiv \lim_{\Delta\to 0} \hat{K}\Delta$, $X^* \equiv \lim_{\Delta\to 0} K\Delta$. Since $\hat{K} \equiv \lfloor \frac{1+A}{1-A}\frac{\delta}{1-\delta} - \frac{\delta}{1+\delta} \rfloor + 1$, then $\hat{X} = \frac{1+A}{r(1-A)}$. Therefore, $K > \hat{X}$ is equivalent to $X^* > \hat{X}$.

Step 1: $X^* > \hat{X}$.

Define $X = \lim_{\Delta\to 0} k\Delta$, $y(X) = \lim_{\Delta\to 0} y_{k\Delta}$. Following Step 2 of Lemma D.4, by D.1 and D.2,

$$(a(X), y(X)) = \begin{cases} (0,0) & 0 \le X \le X^* - \hat{X}. \\ (a^*(X), 1 + \frac{r(1-A)}{1+A}(X - X^*)) & X^* - \hat{X} \le X \le X^*. \\ (0,1) & X \ge X^* \end{cases}$$

Step 2: $X^* \le \hat{X}$ and $X^* \le \frac{1}{r}$.

In the limit, $K \le 2\lfloor \frac{\delta^2}{1-\delta^2} \rfloor$ is equivalent to $X^* \le \frac{1}{r}$. Following Step 2 of Lemma D.5, by D.3, for all $0 \le X \le X^*$,

$$(a(X), y(X)) = \begin{cases} (1,1) & X = \lim_{\Delta\to 0}(2k+1)\Delta. \\ (a^*(X), \frac{1+A-r(1-A)(X^*-X)}{1+A+r(1-A)(X^*-X)}) & X = \lim_{\Delta\to 0} 2k\Delta. \end{cases}$$

Step 3: $X^* \le \hat{X}$ and $X^* > \frac{1}{r}$.

In the limit, $K > 2\lfloor \frac{\delta^2}{1-\delta^2} \rfloor$ is equivalent to $X^* > \frac{1}{r}$. We follow Step 3 of Lemma D.6. Define $\tilde{X} = \lim_{\Delta\to 0} \tilde{K}\Delta$. Then, $\tilde{X} = X^* - \frac{1}{r}$. By D.4, for $0 \le X \le X^* - \frac{1}{r}$,

$$(a(X), y(X)) = \begin{cases} (a^*(X), \frac{(1+A)-r(1-A)(X^*-X)}{2A}) & X = \lim_{\Delta\to 0}(2k+1)\Delta. \\ (a^*(X), \frac{(1+A)-r(1-A)(X^*-X)}{2}) & X = \lim_{\Delta\to 0} 2k\Delta. \end{cases}$$

In the limit $\Delta \to 0$, $y_{\tilde{K}} \to 1$ and $y_{\tilde{K}-1} = \frac{1}{A}y_0 + \frac{(1-\delta)(1-A)}{\delta A}(\tilde{K}-1) = \frac{y_{\tilde{K}}}{A}$. Therefore, $y_{\tilde{K}-1} \to A$. By $D.5$, for $X^* - \frac{1}{r} < X \le X^*$,

$$(a(X), y(X)) = \begin{cases} (1,1) & X = \lim_{\Delta\to 0}(2k+1)\Delta. \\ (a^*(X), \frac{1+A-r(1-A)(X^*-X)}{1+A+r(1-A)(X^*-X)}) & X = \lim_{\Delta\to 0} 2k\Delta. \end{cases}$$

□

Appendix E. Proofs for Section 4.3

Proof of Theorem 4.6:

Proof. By Theorem 3.1 and Assumption 4.9, if the firm only has binary choices I_{i^*} and I_0, then the stationary Markov equilibrium can be characterized by a reputation-building stage $0 < X < X^*$ and a reputation-exploitation stage $X \ge X^*$. Based on this equilibrium, we construct an equilibrium if there are multiple investment choices.

If $0 < X < X^*$, we focus on equilibira in which the buyers play mixed strategies: $y(X) \in (0,1)$. Firstly, the firm will put a probability between 0 and 1 on I_0. Otherwise, the buyers will strictly prefer to buy: $y(X) = 1$, a contradiction. Secondly, by the definition of i^*, $(1-\delta)g_1(I_i, B)y(X) + \delta((1-q_i)V(X+1) + q_iV(X-1)) < (1-\delta)g_1(I_{i^*}, B)y(X) + \delta((1-q_{i^*})V(X+1) + q_{i^*}V(X-1)) = (1-\delta)g_1(I_0, B)y(X) + \delta((1-q_0)V(X+1) + q_0V(X-1))$. Therefore, the firm only mixes between I_{i^*} and I_0.

If $X \geq X^*$, then the buyers buy for sure: $y(X) = 1$. By the definition of i^*, $(1-\delta)g_1(I_i, B) + \delta((1-q_i)V(X+1) + q_iV(X-1)) < (1-\delta)g_1(I_{i^*}, B) + \delta((1-q_i)V(X+1) + q_iV(X-1)) < (1-\delta)g_1(I_0, B)y(X) + \delta((1-q_0)V(X+1) + q_0V(X-1))$. Therefore, the firm plays I_0 for sure at $X \geq X^*$. □

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