STOCHASTIC REPUTATION DYNAMICS UNDER DUOPOLY COMPETITION

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ABSTRACT. This paper studies a dynamic duopoly model of reputation-building in which reputations are treated as capital stocks that are influenced by past investment decisions, and have persistent effects on future payoffs. The setting is a discrete-time discounted stochastic game between two long-run firms and a sequence of short-run buyers, where the state variables are the reputations of the two firms. If buyers can only buy from either firm, then the state of competition is captured by the difference of two reputations. There are two types of stationary Markov equilibria. In *catch-up equilibria*, the leader, with a higher market share, exploits reputation by not investing; the follower invests with positive probability and eventually catches up with the leader. In *permanent leadership equilibria*, depending on the initial state of the economy, the economy asymptotically consists either of two firms competing forever or of just one dominant firm. If buyers have an outside option of not buying, duopoly competition is not a major concern if both reputations are lower than a threshold, above which it is a dominant strategy for buyers to buy from either firm. Ignoring the rival's reputation, each firm focuses on building up its own reputation until at least one reputation is built up higher than the threshold, and a duopoly competition begins to take place.

1. INTRODUCTION

A company's reputation is intrinsically a relative concept. Although it is common to think of a particular company as reputable in some absolute sense, it is really the relative reputation in comparison with other companies' reputations that matters for the success of the company. When we say that Toyota makes reliable cars what we have in mind a benchmark for reliability, which is set by our experience with cars with other brands. When a competitor company succeeds in improving its own reputation, the given firm's performance

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is likely to be hurt, even when its own absolute reputation does not change. Yet, the game-theoretic literature on reputation is mostly focused on the case of a single company, and hence ignores the competitive aspects of relative reputation. The purpose of this paper is to provide a tractable dynamic duopoly model where the relative reputation of a firm is crucial for its success. Moreover, and in line with the recent experience of car companies like Toyota, reputation is treated as a capital stock, rather than a "belief" of the customers. Roughly, a company can influence its absolute reputation by investing in it (e.g. by investing in the quality of its products which leads to better experiences of its customers, which in turn builds up the company's "goodwill.") As the other company can also invest in its own reputation, interesting dynamics of relative reputation can emerge as equilibrium outcomes. For instance, the most reputable company may find it optimal to simply milk its reputation for a while, whereas the competitor may find it optimal to try to catch up fast.

More specifically, we are interested in whether the company with higher reputation (the "leader") is able to maintain a *permanent leadership* or the company with lower reputation (the "follower") can *catch up* with the leader. These two questions are determined by two basic forces. On the one hand, the leader has incentives to exploit a high enough relative reputation and corresponding high market share, which leads to a *catch up* scenario and equal market shares. On the other hand, the leader may invest aggressively whenever it is threatened in order to defend its dominant position, which leads to a *permanent leadership* scenario and corresponding extreme market shares. The main contribution of this paper is that it provides a simple framework to explain rich dynamics, including the *catch up* and *permanent leadership* scenarios just described.

As an additional question, we are also interested in the situation that customers have an outside option of not buying from either firm, which in effect turns a firm's absolute reputation into a relevant variable. That is, when both firms have low reputations, the competition between the two firms is less important than attracting customers away from their outside option. As a result, ignoring the rivals' reputation, each firm focuses on building up its own absolute reputation. If one reputation is built up high enough, buyers are convinced to buy from either firm, and we are back to a duopoly situation.

Formally, we analyze a discrete-time discounted stochastic game with two long-run firms, firm 1 and firm 2, and a sequences of short-run buyers. In each period, firm $i = 1, 2$ sells good i. There is a group of potential buyers who decide to buy from firm 1 or firm 2. In each period, firm 1, firm 2 and a short-run buyer play a product-choice game simultaneously. Firm $i = 1, 2$ decides to invest in reputation for good i or not, and the buyer decides to buy good 1 or good 2. We also study a variant of this model in which the buyer has an outside option of buying from neither firm. Period payoffs depend on the current actions of the players and on two firms' reputation stocks, which are the state variables. Reputation stock i evolves according to a transition rule that depends on firm i's investment decisions. The firms discount the future with the same discount factor and take account of the fact that today's efforts affect the future state of competition. Restricting to stationary Markov equilibria, we study reputation dynamics under two different transition rules: *one-step up transition rule* in which investment results in reputation progress by one step and no investment does not change the reputation, and *one-step up-and-down transition rule* in which investment results in reputation progress by one step and no investment leads to depreciation by one step.

We first study a benchmark model where buyers have the choice of not buying any product, and show that it is without loss of generality to study equilibria which depend only on the relative reputation (the leader's lead) instead of absolute reputation levels. Under the two transition rules mentioned above, there exist two types of stationary Markov equilibria: *catch-up equilibria* and *permanent leadership equilibria*.

In *catch-up equilibria*, the firm with lower reputation (the "follower") can eventually catch up the firm with higher reputation (the "leader"). There are two stages. If the lead is sufficiently ahead, the leader dominates the market by attracting all demands. At the same time, the leader loses the incentive to invest, but the follower invests in order to erode the leadership and to recover the market share. If the lead is not so large, then the leader does not invest and the follower invests with positive probability in such a way that buyers are indifferent between buying from either firm. In equilibrium, as the relative reputation goes up, market share of the leader does not grow at a speed fast enough to provide incentives for the leader to invest. On the contrary, market share of the follower is sensitive to reductions of the lead, which provides an incentive for the follower to invest with positive probability. In the long run, the economy converges to an absorbing state in which both firms have the same level of reputation, and neither firm invests.

The second type is *permanent leadership equilibria*. If the firms have the same reputation, each firm invests in order to prevent the other firm from gaining ground and becoming the permanent leader. The result is a "tie", that is, the market is split in all periods. With a slight lead,[1] the leader invests more aggressively than the follower in order to defend its dominant position. This, in turn, leads to a tendency towards *permanent leadership* and extreme market shares. If the leader's lead is larger than two steps, expecting the leader's aggressive behavior with a slight lead, it is impossible for the follower to catch up, so it does not invest. Since what matters is relative reputation, the leader does not invest either, and consequently the leadership will remain constant. Asymptotically, the economy converges to permanent leadership of a single firm. In all, depending on the initial state of the economy, the economy may asymptotically consist either of two firms competing forever or of just one dominant firm.

When the buyers have an outside option of not buying from either firm, we show that both relative reputation and absolute reputations play a role in the equilibria. Under the two transition rules mentioned above, there exists *generalized catch-up equilibria* that display the same catch-up behavior as in *catch-up equilibria* if both reputations are higher than a threshold level, above which not buying is never a best response for the buyer. We call this the catch-up stage. If both reputations are smaller than the threshold, there is a reputation-building stage where both firms deal with the possibility that buyers do not buy from either firm. Both firms invest with positive probability in such a way that buyers randomize over buying good 1, buying good 2 and possibly not buying. In other words, building reputation becomes a top priority relative to fighting for the leadership. Reputation dynamics are different under the different transition rules. Under *one-step up transition rule*, absorbing states will be eventually reached in which two firms have equal reputation and neither firm invests. Under *one-step up-and-down transition rule*, there is no absorbing state. In a catch-up stage, the follower eventually catches up with the leader, and thereafter neither firm invests. As a consequence, both reputations depreciate and enter the reputation-building stage, in which both reputations move stochastically until hitting the threshold level and back to the reputation-building stage. That is, we obtain a reputation cycle.

[1]Under *one-step up transition rule*, the lead is one step. Under *one-step up-and-down transition rule*, the lead is one step and two steps.

1.1. **Literature Review.** There are three papers closely related to this paper: Aoki (1991), Hörner (2004) and Budd et al. (1993). In these papers, two firms engage in R&D competition for an infinite number of periods. In Aoki (1991), only the leading firm sells the product and earns a exogenous monopoly profit which is independent of the size of the lead. The transition rule is deterministic: a firm either makes a costly effort and advances its state of knowledge by one step, or it does not and its state does not change. The equilibrium result supports *permanent leadership*: no firm invests if the lead is larger than one step, thus firms never alternate positions as leaders and followers. Hörner (2004) extends Aoki (1991) by considering non-deterministic transition rules. In Budd et al. (1993), each firm's current profits are endogenously determined by a one-dimensional state variable: the difference between two firms levels of technology.

In this paper, each firm's market share depends on the current state measured by two firm's reputation levels. We model market share by explicitly introducing a short-run buyer in each period whose payoff is determined both by reputation levels as well as two firms' current investment decisions. In each period, market share for each firm is endogenously determined by the short-run buyer's equilibrium choice. If the short-run buyer can only buy from firm 1 or firm 2, the difference between firms' reputation levels is enough to capture of the state of competition. If the buyer can choose not to buy, two-dimensional reputation levels are needed to capture the state of competition since two firms are also competing with buyer's outside options.

With respect to market share dynamics, there is a related literature that studies dynamic oligopoly competition when consumers have switching costs of changing the product that they purchase from period to period, see a survey by Villas-Boas (2015). Firms are faced with a trade-off between today and tomorrow: a higher price brings higher profit today and a low price increases the future customer base. A central question of this literature is whether market shares tend to equalize over time (*catch-up*), or whether a firm may defend a dominant market position persistently (*permanent leadership*). As illustrated by Chen and Rosenthal (1995) and Kovác and Schmidt (2014), dynamics are driven only by strategic interaction between the firms, namely via the use of mixed pricing strategies in the Markov perfect equilibria. This paper shares similar flavors: (1) Not investing saves a short-term cost today and investing builds up future reputation and brings higher future payoff; (2) This paper also explore whether there is *catch-up* or *permanent leadership*; (3) The stationary Markov equilibria also involve the use of mixing investment strategy.

2. Model

We study a discrete-time stochastic game where two long-run players (henceforth firm 1 and 2) play against an infinite sequence of short-run players (henceforth the buyers). Time is discrete and indexed by $t = 0, \Delta, 2\Delta \ldots, \infty$. Δ is the length of each period. In later sections, we will analyze the case where Δ is small and also the limit as $\Delta \to 0$. A buyer who arrives at time t plays a stage-game with two firms, then exits and does not come back. Both firms discount future payoffs by $\delta = e^{-r\Delta}$ and maximize the expected sum of discounted payoffs. Each short-run buyers only cares about the stage-game payoff.

Reputation of firm i is modeled as a state variable X_i. (X_1, X_2) affects the stage-game payoffs of the buyers. Define $\mathscr{X}_\Delta \equiv \{0, \Delta, 2\Delta, \ldots\}$. The state space is $\mathscr{X}_\Delta^2$, which means that the shift of reputation X_i is proportional to the time interval Δ. This captures the idea that reputation building (or milking) is a smooth process if we restrict the maximal steps of reputation shift to be bounded in each period.

The stage game is a modified version of product-choice game in which the buyers' stage-game payoffs depend on firms' reputation. In each period, two firms and the buyer move simultaneously. There are two pure actions for firm i: I_i and NI_i, which represent investing and not investing. There are two pure actions for the buyer: B_1 and B_2, which represent buying from firm 1 and firm 2. The buyer has no outside option of not buying. We will cover the case where not buying is a choice of the buyer in Section 4.

Table 1 is an example of a stage-game payoff matrix that illustrates a product-choice game that we will study. The row player is firm 1 and firm 2 and the column player is the buyer.

Table 1. No Outside Option for Buyers

	B_1	B_2
I_1	$1,\ \lambda + (1-\lambda)X_1$	$0,\ 0$
NI_1	$2,\ -\lambda + (1-\lambda)X_1$	$0,\ 0$
I_2	$0,\ 0$	$1,\ \lambda + (1-\lambda)X_2$
NI_2	$0, 0$	$2,\ -\lambda + (1-\lambda)X_2$

Notice the following properties of firm i's stage-game payoff. (i) Firm i's stage-game payoff is not directly influenced by reputation X_i.[2] (ii) Firm i is better off if the buyer buys from firm i. (iii) It is a dominant strategy for firm i not to invest. The firm's investment cost is 1 if the buyer buys and 0 if the buyer does not buy. Therefore, the expected investment cost is increasing in the buyer's probability of buying from firm i, which is called *submodularity.* (iv) Firm i prefers (I_i, B_i) to (NI_i, NB_i), which means that firm i is better off if committing to investing is possible. Next, we describe four properties of the buyer's stage-game payoff. (i) Buyer i's stage-game payoff is increasing in X_i if buyer i buys, which means that reputation is valuable for buyer i. (ii) Buyer i is better off if each firm invests. (iii) The buyer prefers to buy from firm if the firm invests in good i, and gets the same payoff 0 if the firm does not invest in good i. (iv) If $X_i - X_{-i} \geq \frac{2\lambda}{1-\lambda}$, it is a dominant strategy for the buyer to buy good i. If $X_i - X_{-i} < \frac{2\lambda}{1-\lambda}$, then two firms can invest in such a way that the buyer is indifferent between B_1 and B_2: $a_2(X) - a_1(X) = \frac{1-\lambda}{2\lambda}(X_2 - X_1)$, where $a_i(X)$ is firm i' s probability of investing at state $X = (X_1, X_2)$.

Assumptions 2.1-2.6 make the above statements formal. Firm i's stage-game payoff is $g_i : \{I_i,\ NI_i\} \times \{B_i, NB_i\} \mapsto \mathbb{R}$. The Buyer's stage-game payoff also depends on the state variable $(X_1, X_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. The buyer's stage-game payoff from buying good i is $g_{bi} : \{I_i,\ NI_i\} \times \mathbb{R}^+ \mapsto \mathbb{R}$.

Assumption 2.1: $g_i(NI_i, B_i) \geq g_i(I_i, B_i)$, $g_i(NI_i, B_j) \geq g_1(I_i, B_j)$; $g_i(I_i, B_i) > g_i(I_i, B_j)$, $g_i(NI_i, B_i) > g_i(NI_i, B_j)$; $g_i(I_i, B_i) > g_i(NI_i, B_j)$.

Assumption 2.2: $g_i(NI_i, B_i) - g_i(I_i, B_i) > g_i(NI_i, B_j) - g_i(I_i, B_j)$.

Assumption 2.3 (Symmetry): $g_1(I_1, B_1) = g_2(I_2, B_2)$; $g_1(NI_1, B_1) = g_2(NI_2, B_2)$.

Assumptions 2.1-2.2 describe the stage-game payoff of the firm i. Assumption 2.1 tells us that in a stage-game, firm i prefers not to invest and firm i is better off if the buyer buys from firm i. Moreover, the firm prefers cooperation (I_i, B_i) to non-cooperation (NI_i, NB_i), which means that firm i has an incentive to build reputation. Assumption 2.2 is the *submodularity* of the firm's payoff, which characterizes the conflict between firm i and the buyer.

[2]Justified by several real-world reputation stories as we saw in the introduction, this paper models the impact of reputation on sales instead of prices, which may directly influence the firm's payoff. A model that captures sales and prices is left for future research.

Assumption 2.3 says that firm's stage-game payoffs are symmetric to each other. This is a simplifying assumption.

Assumption 2.4: $g_{bi}(I_i, X_i) > g_{bi}(NI_i, X_i)$ for any X_i.

Assumption 2.5: $g_{bi}(I_i, X_i)$ and $g_{bi}(NI_i, X_i)$ are strictly increasing in X_i.

Assumption 2.6 (Linearity): $g_{bi}(I_i, X_i) = \alpha_1 + \beta X_i$, $g_{bi}(NI_i, X_i) = \alpha_0 + \beta X_i$.

Assumptions 2.4-2.6 describe the stage-game payoff of buyer i. If the buyer buys from firm i, Assumption 2.4 tells us the buyer wants firm i to invest and Assumption 2.5 means that reputation X_i is valuable for the buyer. Assumption 2.6 is for tractability. There exists a threshold $X^* \equiv \frac{\alpha_1 - \alpha_0}{\beta}$, which determines the buyer's choice between B_1 and B_2. Define reputation gap as $X_d = X_2 - X_1$. If $X_d > X^*$, then the buyer buys from firm 2. If $X_d < -X^*$, then the buyer buys from firm 1. Define a_i as mixed strategy of firm i: the probability of playing I_i. If $-X^* \le X_d \le X^*$, there exists a_1 and a_2 to make the buyer indifferent between B_1 and B_2: $a_1(\alpha_1 + \beta X_1) + (1 - a_1)(\alpha_0 + \beta X_1) = a_2(\alpha_1 + \beta X_2) + (1 - a_2)(\alpha_0 + \beta X_2)$, which implies that $a_1 - a_2 = \frac{\beta}{\alpha_1 - \alpha_0} X_d = \frac{X_d}{X^*} \le 1$.

Let $a_i \in [0, 1]$ denote the mixed strategy of firm i: the probability of playing I_i. Let $y_i \in [0, 1]$ denote the mixed strategy of the buyer: the probability of playing B_i. Observe that $y_1 + y_2 = 1$. Reputation (X_1, X_2) only has an impact on the buyer's payoff. Define $g_i(a_i, y_i)$ as the expected stage payoffs of firm i: $g_i(a_i, y_i) = g_i(I_i, B_i)a_i y_i + g_i(NI_i, B_i)(1 - a_i)y_i + g_i(I_i, NB_i)a_i(1 - y_i) + g_i(NI_i, NB_i)(1 - a_i)(1 - y_i)$. Denote $g_b(a_1, a_2, y_1, y_2, X_1, X_2)$ as the expected stage payoffs of the buyer at state (X_1, X_2).

$$g_b(a_1, a_2, y_1, y_2, X_1, X_2) = \sum_{i=1}^{2} \big(g_{bi}(I_i, X_i)a_i y_i + g_{bi}(NI_i, X_i)(1 - a_i)y_i\big).$$

Finally, we specify the transition rules of state variable (X_1, X_2), which characterize how the current actions have a persistent impact on the future buyers' payoffs. We consider Markov transition rules represented by two transition probabilities P_i for reputation X_i:

$$P_i : \{I_i, NI_i\} \times \mathscr{X}_\Delta \mapsto \Delta(\mathscr{X}_\Delta).$$

Given firm i's action $f_i \in \{I_i, NI_i\}$ and the current state X_i, $P_i(f_i, X_i)$ is the probability of the state X_i' in the next period. Given the firm i's mixed strategy a_i and the current state X_i, the probability of next state X_i' is $P(a_i, X_i) = a_i P(I_i, X_i) + (1 - a_i)P(NI_i, X_i)$.

3. Equilibrium Analysis

We consider *stationary Markov equilibria* in which two firms and the buyers play *stationary Markov strategies.* Denote $(a_1(X), a_2(X), y_1(X), y_2(X))$ as the mixed actions of two firms and the buyers which only depend on the current state X. Define $V_i(X)$ as firm i's continuation value at state X.[3]

Definition 3.1. $(a_i(X), y_i(X), V_i(X))$ is a *stationary Markov Equilibrium* if for all $X = (X_1, X_2)$, $i = 1, 2$, $P_i = P(a_i, X_i)$, we have

$$\begin{aligned} V_i(X) &= \max_{a_i \in [0,1]} (1-\delta) g_i(a_i, y_i(X)) + \delta E_{P_1,P_2} V(X'). \\ a_i(X) &\in \operatorname*{argmax}_{a_i \in [0,1]} (1-\delta) g_i(a_i, y_i(X)) + \delta E_{P_1,P_2} V(X'). \\ (y_1(X), y_2(X)) &\in \operatorname*{argmax}_{(y_1, 1-y_1) \in [0,1]^2} g_b(a_1(X), a_2(X), y_1, 1-y_1, X). \end{aligned}$$

3.1. **One-Step Up Transition Rule.**

In this section, we study *one-step up transition rule* as below:

$$P(X_i' = X_i + \Delta | I_i) = 1,\ P(X_i' = X_i | NI_i) = 1,$$

where X_i is firm i's current reputation and X_i' is firm i's reputation in next period. If firm i invests, reputation X_i increases by one-step. Otherwise, X_i remains the same. Observe that firm j and the buyer's action do not influence the transition of reputation X_i.

Define two parameters A and γ which capture firm i's payoff structure: for $i = 1, 2$,

$$A_i = \frac{g_i(1,1) - g_i(1,0)}{g_i(0,1) - g_i(0,0)}, \gamma_i = \frac{g_i(0,0) - g_i(1,0)}{g_i(0,1) - g_i(0,0)}.$$

The parameter A_i captures the *submodularity* of firm i's payoffs. Higher A_i means a low degree of submodularity, thus a higher intensity of conflict between firm i and the buyers. The parameter γ_i captures firm i's investment cost if the buyer does not buy. By Assumption 2.3 (symmetry), $A_1 = A_2 \equiv A$ and $\gamma_1 = \gamma_2 \equiv \gamma$. By Assumptions 2.1 and 2.2, $\gamma < A < 1$.

Define reputation difference $X_d \equiv X_2 - X_1$. We construct the following *catch-up equilibrium I*, which is only determined by X_d. By symmetry, it is enough to consider $X_d \geq 0$.

(1) If $X_d > X^*$, then $a_1(X_d) = 1$, $a_2(X_d) = 0$, $y_1(X_d) = 0$ and $y_2(X_d) = 1$.

[3] $V_i(X)$ is bounded above by $g_i(NI_i, B_i)$, which is the highest stage payoff that firm i can get, so the transversality condition is satisfied.

(2) If $0 \le X_d < X^*$, then $a_1(X_d) = \frac{X_d}{X^*}$, $a_2(X_d) = 0$, $y_1(X_d) = (\frac{1}{2} + \frac{\gamma}{1-A})e^{-r(1-A)X_d} - \frac{\gamma}{1-A}$ and $y_2(X_d) = 1 - y_1(X_d)$.

Define $\bar{X}_\gamma$ as the solution to $\frac{r}{2}(X)^{1-rX}\int_0^X e^{-r(1-A)x}x^{rX-1}dx = \frac{1-A+\frac{2-A}{1-A}\gamma}{1+\frac{2\gamma}{1-A}}$.

Theorem 3.2. *Under one-step up transition rule, $X^* \in (\bar{X}_\gamma, \frac{1}{r(1-A)}\ln(\frac{1-A+2\gamma}{2\gamma}))$ and $\Delta \to 0$, there exists a limiting equilibrium characterized by the catch-up equilibrium I.*

Under the assumption that buyers have no outside option of not buying, Theorem 3.2, proved in the Appendix A, asserts that state of competition is completely captured by reputation difference (firm's 2's lead) $X_d \equiv X_2 - X_1$, and there exists a *catch-up equilibrium I* if the threshold X^*, beyond which the leader dominates the market share, satisfies some condition.

Catch-up equilibrium I describes the behavior of reputation-exploitation by the leader as well as reputation-building by the follower. Let us now provide some intuition on why this is an equilibrium.

As $X_d > X^*$, the reputation lead is so large that market share totally belongs to the leader. The leader does not invest for two reasons: (1) By *submodularity* of firms' payoff, with full market share, the leader incurs large investment cost; (2) Market share of the leader is insensitive to the change of X_d, thus the leader's future benefit of building reputation is small; (3) If $X^* > \bar{X}_\gamma$, then we can show that this future benefit is indeed less than the short-term investment cost. Since the leader's continuation value function is a concave function of X_d for $X_d > X^*$, higher X^* implies that the value function is less sensitive to the increase of X_d for any $X_d > X^*$. This, in turn, implies a lower future benefit of building reputation. We can construct an equilibrium in which the follower invests for sure for two reasons: (1) By *submodularity* of firms' payoff, without any market share, the follower incurs small investment cost; (2) The follower has the chance of eroding the reputation lead due to the fact that the leader does not invest.

As $0 \le X_d < X^*$, two firms share the market demand according to the relative reputation. In equilibrium, the market share is constructed to provide "proper" incentive for the follower to catch up with the leader. We check the condition under which the follower invests with positive probability and the leader does not invest: (1) Market share of the follower $y_1(X_d)$ is decreasing in X_d in a convex way so that the follower makes mixing investment strategy. (2)

$X^* < \frac{1}{r(1-A)} \ln(\frac{1-A+2\gamma}{2\gamma})$ guarantees that $y_1(X^*-) > 0$. This condition holds if γ, investment cost without market share, is small enough.[4] Intuitively, small investment cost is necessary for the follower to invest. (3) $y_2(X_d) = 1 - y_1(X_d)$ is increasing and concave in X_d but not at a speed high enough for the leader to invest for $X_d < X^*$. (4) Notice that $y_1(X_d)$ is not continuous at $X_d = X^*$: $y_1(X^*-) > 0 = y_1(X^*+)$. In all, given the fact that the leader does not invest and the "proper" dynamics of the market share, the follower invests with positive probability to catch up with the leader. Given the follower's reputation-building behavior and dynamics of the market share, the leader does not invest since its market share is not sensitive enough to the increase of reputation lead.

In *catch-up equilibrium I*, the lead is eroded by the follower's behavior of catching-up. Eventually, two firm's reputations are tied: $X_d = 0$ and each firm gains half of the market share. Expecting the catch-up behavior of the future follower, each firm loses the incentive to invest, thus the state of a tie is an absorbing state. To be specific, the dynamics of reputation is as follows: starting from $X_2 = X^* > X_1$, two reputations reach $X_1 = X_2 = X^*$ eventually and stay there forever.

We can generate the result to the situation where two firm are asymmetric. Assume WLOG that two firms have different discount factors: $r_1 \neq r_2$. For simplicity, assume that $\gamma = 0$. Qualitatively similar to the *catch-up equilibria I*, the following *asymmetric catch-up equilibria I* also displays the behavior of catching up.

(1) If $X_d \geq X^*$, then $a_1(X_d) = 1$, $a_2(X_d) = 0$, $y_1(X_d) = 0$ and $y_2(X_d) = 1$.
(2) If $0 \leq X_d < X^*$, then $a_1(X_d) = \frac{X_d}{X^*}$, $a_2(X_d) = 0$, $y_1(X_d) = y_1(0)e^{-r_1(1-A)X_d}$, $y_2(X_d) = 1 - y_1(X_d)$.
(3) If $-X^* < X_d < 0$, then $a_2(X_d) = \frac{-X_d}{X^*}$, $a_1(X_d) = 0$, $y_2(X_d) = y_2(0)e^{r_2(1-A)X_d}$, $y_1(X_d) = 1 - y_2(X_d)$.
(4) If $X_d \leq -X^*$, then $a_2(X_d) = 1$, $a_1(X_d) = 0$, $y_2(X_d) = 0$ and $y_1(X_d) = 1$.

Define $\bar{X}_a$ as the solution to $X^{1-r_2X} \int_0^X e^{-r_1(1-A)x} x^{r_2X-1} dx + X^{1-r_1X} \int_0^X e^{-r_2(1-A)x} x^{r_1X-1} dx = (\frac{1}{r_1} + \frac{1}{r_2})(1-A)$.

Theorem 3.3. *Under one-step up transition rule, $X^* > \bar{X}_a$, $r_1 \neq r_2$ and $\Delta \to 0$, there exists a limiting equilibrium characterized by asymmetric catch-up equilibrium I.*

[4]As $\gamma \to 0$, this condition always holds.

Besides *catch-up equilibria I*, we find that there always exists another type of equilibria with fixed time interval Δ and a given discount factor δ: *permanent leadership equilibria I* in which the follower cannot catch up with the leader. For notation simplicity, denote $a_i(k)$ and $y_i(k)$ as $a_i(k\Delta)$ and $y_i(k\Delta)$, where $k\Delta$ is reputation difference $X_2 - X_1$. The equilibria are described as below:

(1) $a_1(0) = a_2(0) = 1$ and $y_1(0) = y_2(0) = \frac{1}{2}$.
(2) $1 \geq a_2(1) > a_1(1) > 0$ and $y_1(1) = 0$, $y_2(1) = 1$.
(3) $a_1(k) = a_2(k) = 0$ and $y_1(k) = 0$, $y_2(k) = 1$ for $k \geq 2$.

Theorem 3.4. *Under one-step up transition rule, $\gamma < \frac{A}{2}$ and δ is large enough, there exists a permanent leadership equilibrium I.*

Theorem 3.4 constructs a *permanent leadership equilibrium I* in which both firms lose incentives to invest if the leadership is larger than one step.

If $\gamma < \frac{A}{2}$, each firm's benefit of splitting the market is larger than the corresponding investment cost, thus each firm earns a positive profit. In equilibrium, both firms invest when they are even: $X_d = 0$, preventing the rival from gaining the leadership. The value of catching up is less attractive for the follower and it will drop out even when the lead is small. In fact, investment continues only when the follower is one step behind.

If the leader is one-step ahead, both firms are indifferent between investing and not investing, and the buyer buys from the leader. First, the follower invests with probability less than one. Otherwise, the leader invests as a best response to the follower investing. This, in turn, ruins the follower's hope of catching-up, a contradiction. As a result, the continuation value for the follower is zero when it is one-step behind since with positive probability, the follower chooses not to invest and gives up any chance of regaining positive market share in the future. Second, the follower invests with probability larger than zero. Otherwise, the leader's best response is not to invest, which gives the follower high incentive to tie the reputation by investing, a contradiction. Third, the leader invests with probability less than one. Otherwise, the follower's best response is not to invest, since even investing will not tie the reputation. As a result, the leader strictly prefers not to invest and consequently the follower invests for sure, a contradiction. Moreover, we can show that the leader invests with higher probability than the follower, leading to the leader's domination of market share. As the follower is two or more steps behind, the value of catching-up is zero and he will not choose to invest-at two or more steps behind. Therefore, there is no need for the leader to invest.

Depending on the initial state of the economy, the economy may asymptotically consist either of two firms competing forever or of just one dominant firm. If the initial state is a tie, both firms invest and split the market share. Therefore, a tie is an absorbing state. If the initial state is more than one-step, neither firm invests, the initial state is absorbing and the leader gain a monopoly profit forever. If the initial state is one-step, the lead goes up and down stochastically.

3.2. **One-step up-and-down transition rule.** In this section, we consider an *one-step up-and-down transition rule* as below:

$$P(X_i' = X_i + \Delta | I_i) = 1, \; P(X_i' = \max\{0, X_i - \Delta\} | NI_i) = 1,$$

where X_i is firm i's current reputation and X_i' is firm i's reputation in next period. If firm i invests, reputation X_i goes up by one-step. Otherwise, X_i goes down by one-step. Observe that firm j and the buyer's actions do not influence the transition of reputation X_i. For simplicity, we assume that $\gamma = 0$.

We study the following *catch-up equilibrium II*, which only depends on $X_d \equiv X_2 - X_1$.

(1) If $X_d \geq X^*$, then $a_1(X_d) = 1$, $a_2(X_d) = 0$, $y_1(X_d) = 0$ and $y_2(X_d) = 1$.
(2) If $0 < X_d < X^*$, then $a_1(X_d) = \frac{X_d}{X^*}$, $a_2(X_d) = 0$, $y_1(X_d) = \frac{1}{2}e^{\frac{-r(1-A)}{2}X_d}$ and $y_2(X_d) = 1 - y_1(X_d)$.

Define $\tilde{X}$ as the solution to $\frac{rX}{4}X^{-\frac{r}{2}X}\int_0^X e^{-\frac{r}{2}(1-A)x}x^{\frac{r}{2}X-1}dx = 1 - A$.

Theorem 3.5. *Under one-step up-and-down transition rule and $X^* > \tilde{X}$, then there exists a catch-up equilibrium II.*

We will show that there always exists the following *permanent leadership equilibrium II*: For notation simplicity, denote $a_i(k)$ and $y_i(k)$ as $a_i(k\Delta)$ and $y_i(k\Delta)$, where $k\Delta$ is reputation difference $X_2 - X_1$. The equlibrium only depends on k:[5]

(1) $a_1(0) = a_2(0) = 1$ and $y_1(0) = y_2(0) = \frac{1}{2}$.

[5]In Appendix, we show that $a_i(X_1, X_2)$ under $X_1 = 0$ and $X_2 - X_1 = k\Delta$ is different from that under $X_1 \geq 1$ and $X_2 - X_1 = k\Delta$, but $y_i(X_1, X_2)$ remains the same.

(2) $1 = a_2(k) \geq a_1(k) > 0$ and $y_1(k) = 0$, $y_2(k) = 1$ for $k = 1, 2$.

(3) $a_1(k) = a_2(k) = 0$ and $y_1(k) = 0$, $y_2(k) = 1$ for $k \geq 3$.

Theorem 3.6. *Under one-step up-and-down transition rule, $\gamma = 0$ and δ is large enough, there exists a permanent leadership equilibrium II.*

Theorem 3.6 constructs a *permanent leadership equilibrium II* in which both firms do not invest if the leadership is more than two steps. Compared with *permanent leadership equilibrium I*, investment continues if the follower is one step behind or two steps behind. The intuition is as follows: expecting to catch up with the leader with a jump of two steps, the follower has incentive to invest if it is only two steps behind. However, if the lead is more than two steps, the follower loses incentives to invest and it is a best response for the leader not to invest. If there is a tie, both firms invest and equally split the market share. If the lead is one-step, the leader may lose its leadership since "leapfroging" is possible if the leader does not invest and the follower invests.

4. Extensions: Outside Option for the Buyers

In this section, we study another version of the basic model in previous sections. In each period, instead of only two choice B_1 and B_2, the buyer has three choices: B_1, B_2 and NB, which represent buying from firm 1, buying from firm 2 and not buying.

Table 2 is an example of a stage-game payoff matrix that illustrates a product-choice game that we will study. The row player is firm i and the column player is the buyer.

Table 2. Outside Option for Buyers

	B_1	B_2	NB
I_1	$1,\ \lambda + (1-\lambda)X_1$	$0,\ 0$	$0,\ 0$
NI_1	$2,\ -\lambda + (1-\lambda)X_1$	$0,\ 0$	$0,\ 0$
I_2	$0,\ 0$	$1,\ \lambda + (1-\lambda)X_2$	$0,\ 0$
NI_2	$0, 0$	$2,\ -\lambda + (1-\lambda)X_2$	$0,\ 0$

Assumption 4.1: $g_i(I_i, B_j) = g_i(I_i, NB)$, $g_i(NI_i, B_j) = g_i(NI_i, NB)$.

Assumption 4.2: The buyer gets zero payoff if the buyer does not buy.

Assumption 4.3 (Linearity): $g_{bi}(I_i, B_i, X_i) = \alpha_1 + \beta X_i$, $g_{bi}(NI_i, B_i, X_i) = \alpha_0 + \beta X_i$, where $\alpha_1 > 0 > \alpha_0$, $\beta > 0$.

Assumptions 4.1 says that firm i gets the same payoff if the buyer does not buy from firm i. Assumption 4.3 is for tractability. There are two thresholds $X^* \equiv \frac{\alpha_1 - \alpha_0}{\beta}$ and $X^{**} \equiv \frac{-\alpha_0}{\beta}$.

The threshold X^* determines the choice between B_1 and B_2. Define $X_d = X_2 - X_1$. If $X_d > X^*$, then B_2 dominates B_1. If $X_d < -X^*$, then B_1 dominates B_2. If $-X^* \leq_d X \leq X^*$, there exists a_1 and a_2 such that the buyer is indifferent between B_1 and B_2. Consider the condition under which the buyer is indifferent between B_1 and B_2: $a_1(\alpha_1 + \beta X_1) + (1 - a_1)(\alpha_0 + \beta X_1) = a_2(\alpha_1 + \beta X_2) + (1 - a_2)(\alpha_0 + \beta X_2)$, which implies that $a_1 - a_2 = \frac{\beta}{\alpha_1 - \alpha_0} X_d$.

The threshold X^{**} determines the choice between B_i and NB. If $X_i \geq X^{**}$, then B_i weakly dominates NB. If $X_i < X^{**}$, then there exists $a_i^*(X_i) \in (0, 1)$ such that if $a_i = a_i^*(X_i)$, the buyer is indifferent between B_i and NB. It is trivial that $a_i^*(X_i) = \frac{X^{**} - X_i}{X^*}$.

Let $a_i \in [0, 1]$ denote the mixed strategy of firm i: the probability of playing I_i. Denote $y_i \in [0, 1]$ as the probability of buying from firm i: B_i. Notice that $y_1 + y_2 \in [0, 1]$ and $1 - y_1 - y_2 \in [0, 1]$ is the probability of playing NB. Reputation (X_1, X_2) has an impact only on the payoffs of the buyers. Moreover, firm j's investment decision does not influence the payoff of firm i. Denote $g_i(a_i, y_i)$ as the expected stage payoffs of firm i:

$$g_i(a_i, y_i) = g_i(I_i, B_i)a_i y_i + g_i(NI_i, B_i)(1-a_i)y_i + g_i(I_i, NB)a_i(1-y_i) + g_i(NI_i, NB)(1-a_i)(1-y_i).$$

Let $g_b(a_1, a_2, y_1, y_2, X_1, X_2)$ denote the expected stage payoffs of the buyer at state (X_1, X_2).

$$g_b(a_1, a_2, y_1, y_2, X_1, X_2) = \sum_{i=1}^{2} \big(g_{bi}(I_i, X_i)a_i y_i + g_{bi}(NI_i, X_i)(1 - a_i)y_i\big).$$

4.1. **One-Step Up Transition Rule.** In this section, we study the *one-step up transition rule* defined in Section 3.1. Assume that $\gamma = 0$ for simplicity. Define $X = (X_1, X_2)$ as the state variable. WLOG, we focus on the case $X_2 \geq X_1$.

We consider the following *generalized catch-up equilibria I*:

(1) If $X \notin [0, X^{**}] \times [0, X^{**}]$, then there is a catch-up stage.
   (a) If $X_2 - X_1 \geq X^*$, then $a_1(X) = 1$, $a_2(X) = 0$, $y_1(X) = 0$ and $y_2(X) = 1$.
   (b) If $0 < X_2 - X_1 < X^*$, then $a_1(X) = \frac{X_2 - X_1}{X^*}$, $a_2(X) = 0$, $y_1(X) \in (0, 1)$, $y_2(X) = 1 - y_1(X)$.

(2) If $X \in [0, X^{**}] \times [0, X^{**}]$, then there exists a reputation-building stage. $a_i(X) = \frac{X^{**} - X_i}{X^*}$, and buyers are indifferent among B_1, B_2 and NB.

Define $\bar{X}$ as the solution to $\frac{r}{2}(X)^{1-rX} \int_0^X e^{-r(1-A)x} x^{rX-1} dx = 1 - A$.

Theorem 4.1. *Under one-step up transition rule and $X^* > \bar{X}$, then there exists a generalized catch-up equilibrium I.*

Theorem 4.1 characterizes *generalized catch-up equilibria I* under *one-step up transition rule.* There are two stages: a reputation-building stage and a catch-up stage. In the catch-up stage where at least one of the reputations are higher than a threshold level X^*, above which not buying is never a best response for the buyer, the follower eventually catches up with the leader. Buyers mix between buying good 1 and good 2; the leader has no incentive to invest; the follower invests with positive probability if the lead is small and invests for sure if the lead is large enough. Eventually, absorbing states are reached in which two firm have equal reputation and neither firm invests.

In the reputation-building stage where both reputations are smaller than the threshold X^*, both firm build reputation without fighting for the leadership since buyers may be not buy from either firm. Both firms invest with positive probability in such a way that buyers randomize over buying good 1, buying good 2 and not buying. In the long-run, there is a unique absorbing state in which two firms have the same reputation which is exactly the threshold level X^*.

Besides *generalized catch-up equilibria I*, we will show that there always exist other equilibria in which the leader can guarantee its dominant position. We consider the following *generalized permanent leadership equilibria I*:

(1) If $X \notin [0, X^{**}] \times [0, X^{**}]$, then the equilibria only depend on $X_2 - X_1$.
   (a) If $X_2 - X_1 = 0$, then $a_1(X) = a_2(X) = 1$ and $y_1(X) = y_2(X) = \frac{1}{2}$.
   (b) If $X_2 - X_1 = \Delta$, then $1 \geq a_2(X) > a_1(X) > 0$ and $y_1(X) = 0$, $y_2(X) = 1$.
   (c) If $X_2 - X_1 \geq 2\Delta$, then $a_1(X) = a_2(X) = 0$ and $y_1(X) = 0$, $y_2(X) = 1$.

(2) $X \in [0, X^{**}] \times [0, X^{**}]$.
   (a) If $X_2 - X_1 = 0$, then $a_1(x) = a_2(X) = 1$ and $y_1(X) = y_2(X) = \frac{1}{2}$.
   (b) If $X_2 - X_1 = \Delta$, then $a_1(X) \in (0,1)$, $a_2(X) \in (0,1)$ and $y_1(X) = 0$, $y_2(X) \in (0,1)$.

(c) If $X_2 - X_1 \geq 2\Delta$, then $a_1(X) = 0$, $a_2(X) = \frac{X^{**}-X_2}{X^*}$ and $y_1(X) = 0$, $y_2(X) \in (0,1)$.

Theorem 4.2. *Under one-step up transition rule and $\gamma < \frac{A}{2}$ and δ is large enough, there exists a generalized permanent leadership equilibrium I.*

Theorem 4.2 describes how the leader maintains its leadership permanently. If there is a tie: $X_1 = X_2$, both firms invest and equally split the market. If the initial state is more than one-step, the leader can defend its leadership forever. For $X \notin [0, X^{**}] \times [0, X^{**}]$, not buying is a dominated strategy for the buyers, thus the equilibria only depend on $X_2 - X_1$. In equilibrium, if the follower is more than one-step behind, neither firm invests and the leader dominates the market. The intuition is the same as under *permanent leadership equilibrium I*. For $X \in [0, X^{**}] \times [0, X^{**}]$, instead of defending the leading position, the leader focuses on building its reputation since it is possible that buyers do not buy from the leader. In equilibrium, the leader mixes between investing and not investing and buyers randomize over buying from the leader or not buying. If the leader's reputation hits the threshold X^*, the leader begins to exploit the reputation by not investing. In all, there is reputation cycle for the leader i: reputation-building for $X_i < X^*$ and reputation exploitation for $X_i > X^*$ and the follower drops out of the market.

4.2. **One-step up-and-down transition rule.** In this section, we consider an *one-step up-and-down transition rule* defined in Section 3.2.

We consider the following *generalized catch-up equilibria II*:

(1) If $X \notin [0, X^{**}] \times [0, X^{**}]$, then there exists a catch-up stage.
   (a) If $X_2 - X_1 \geq X^*$, then $a_1(X) = 1$, $a_2(X) = 0$, $y_1(X) = 0$ and $y_2(X) = 1$.
   (b) If $0 < X_2 - X_1 < X^*$, then $a_1(X) = \frac{X_2-X_1}{X^*}$, $a_2(X) = 0$, $y_1(X) \in (0,1)$, $y_2(X) = 1 - y_1(X)$.
(2) If $X \in [0, X^{**}] \times [0, X^{**}]$ then there exists an increasing function $f(X_1, X_2)$ such that
   (a) If $f(X_1, X_2) \leq 0$, then there exists a reputation-building stage I: $a_i(X) = \frac{X^{**}-X_i}{X^*}$, and the buyer is indifferent among B_1, B_2 and NB.
   (b) If $f(X_1, X_2) \geq 0$, then there exists a reputation-building stage II: $a_i(X) \geq \frac{X^{**}-X_i}{X^*}$ and $a_1(X) - a_2(X) = \frac{X_2-X_1}{X^*}$, and buyers are indifferent among B_1, B_2.

Theorem 4.3. *Under one-step up-and-down transition rule and $X^* > \bar{X}$, then there exists a generalized catch-up equilibrium II.*

Theorem 4.3 characterizes *generalized catch-up equilibria II* under *one-step up-and-down transition rule*. There are three stages: a reputation-building stage I, a reputation-building stage II and a catch-up stage. The equilibrium behavior in the catch-up stage is similar as that under *one-step up transition rule*. If both reputations are smaller than the threshold X^*, it is not only the relative reputation $X_2 - X_1$ that determines the equilibria. There are two reputation-building stages.

For very low reputation levels: $f(X_1, X_2) \leq 0$, there is a reputation-building stage I. Each firm is concerned about convincing buyers to buy its product instead of attracting buyers from the rivals. In equilibrium, buyers randomize over buying good 1, buying good 2 and not buying and each firm builds its own reputation, ignoring the rival's reputation.

For intermediate reputation levels: $f(X_1, X_2) \geq 0$, there is a reputation-building stage II, which display the features of both reputation-building stage I and catch-up stage. Two firms compete for buyers in such a way that buyers randomize over buying good 1 and buying good 2: $a_1(X) - a_2(X) = \frac{X_2 - X_1}{X^*}$. Moreover, each firm invests with a probability high enough that buyers never choose not to buy: $a_i(X) > a_i^*(X_i) \equiv \frac{X^{**} - X_i}{X^*}$. In all, each firm's strategy depends on both reputation levels and buyers randomize between buying good 1 and buying good 2. In the long-run, there is no absorbing state and each reputation moves stochastically until hitting the threshold X^* and going back to the reputation-building stage. In all, starting from lower reputations, both reputations cycle in the reputation-building stage.

5. Conclusion

This paper has developed a model in which two firms dynamically compete for market shares by building reputations, which are treated as capital stocks that are influenced by past investment decisions, and have persistent effects on future payoffs. If buyers can only buy from either firm, the state of competition is captured by the difference of two reputations. There are two types of stationary Markov equilibria. In *catch-up equilibria*, the leader, with a higher market share, is eventually caught up by the followers. In *permanent leadership equilibria*, the economy asymptotically consists either of two firms competing forever or of just one dominant firm. If buyers have an outside option of not buying, duopoly competition is not a major concern if both reputations are lower enough. Ignoring the rival's reputation, each firm focuses on building up its own reputation until at least one reputation is built up higher than a certain level, and a duopoly competition begins to take place.

Appendix A. Proofs of Section 3.1

In this section, we prove the results under *one-step up transition rule.*

Proof of Theorem 3.2:

Proof. In this section, denote $X \equiv X_d = X_2 - X_1$ for notational simplicity.

Step 1: $X > X^*$.

For $X > X^*$, the optimality condition implies that $V_1'(X) \leq -r\gamma$ and $V_2'(X) \leq r(1-A+\gamma)$. The value functions satisfy $V_1'(X) = -r(V_1(X)+\gamma)$ and $V_2'(X) = r(1-V_2(X))$. Therefore, $V_2(X) \geq A-\gamma$ and $V_1(X) \geq 0$. Specifically, $V_2(X^*) = A-\gamma$.

Step 2: $0 < X < X^*$.

Consider the optimality condition of firm 1: $V_1'(X) = -r[(1-A)y_1(X)+\gamma]$ and $V_1(X) = y_1(X)$. Then, we can solve for $y_1(X)$ and $y_2(X)$, together by $y_1(0) = y_2(0) = \frac{1}{2}$:

$$
\begin{aligned}
y_1(X) &= (\frac{1}{2} + \frac{\gamma}{1-A})e^{-r(1-A)X} - \frac{\gamma}{1-A}.\\
y_2(X) &= 1 + \frac{\gamma}{1-A} - (\frac{1}{2} + \frac{\gamma}{1-A})e^{-r(1-A)X}.
\end{aligned}
$$

Furthermore, $a_1(X) = \frac{X}{X^*}$ so that $y_1 \in (0,1)$.

For $0 < X < X^*$, we consider the optimality condition of firm 2: $a_2(X) = 0$, $V_2'(X) < r((1-A)y_2(X)+\gamma)$ and $rV_2(X) = ry_2(X) - a_1V_2'(X) = ry_2(X) - \frac{X}{X^*}V_2'(X)$. Therefore, we can solve $V_2(X)$ by checking the following first order ODE:

$$V_2'(X) = \frac{rX^*}{X}(1 + \frac{\gamma}{1-A} - (\frac{1}{2} + \frac{\gamma}{1-A})e^{-r(1-A)X} - V_2(X)).$$

For $0 < X < X^*$, there is a unique solution:

$$V_2(X) = 1 + \frac{\gamma}{1-A} - \frac{rX^*}{2}(1 + \frac{2\gamma}{1-A})X^{-rX^*}\int_0^X e^{-b(1-A)x}x^{rX^*-1}dx.$$

Step 3: Check the optimality condition: $V_2(X^*) \geq A-\gamma$ and $V_2'(X) < r((1-A)y_2(X)+\gamma)$ for $0 < X < X^*$. Moreover, $y_2(X^*) \leq 1$ implies that $X^* \leq \frac{1}{r(1-A)}\ln(\frac{1-A+2\gamma}{2\gamma})$.

In order to show $V_2(X^*) \geq A-\gamma$, we need

$$h(X^*) \equiv \frac{r}{2}(X^*)^{1-rX^*}\int_0^{X^*} e^{-r(1-A)x}x^{rX^*-1}dx \leq \frac{1-A+\frac{2-A}{1-A}\gamma}{1+\frac{2\gamma}{1-A}}.$$

Since $h(X^*)$ is decreasing in X^*, then the above inequality is equivalent to $X^* \geq \bar{X}_\gamma$. Furthermore, It is trivial to show that RHS is increasing in γ, thus $\bar{X}_\gamma$ is decreasing in γ.

Take derivative of $rV_2(X) = ry_2(X) - \frac{X}{X^*}V_2'(X)$, we have $(r + \frac{1}{X^*})V_2'(X) + \frac{X}{X^*}V_2''(X) = by_2'(X)$. At $X = 0$, $V_2'(0) = \frac{r^2(1-A)(\frac{1}{2}+\frac{\gamma}{1-A})}{r+\frac{1}{X^*}} < r(\frac{1-A}{2} + \gamma) = r((1-A)y_2(0) + \gamma)$, which satisfy the optimality condition. Furthermore, $V_2''(0) = -r^3(1-A^2)(\frac{1}{2}+\frac{\gamma}{1-A})(r+\frac{2}{X^*})^{-1} < 0$. Therefore, $V_2'(X) < b((1-A)y_2(X) + \gamma)$ at $X \in (0, \epsilon)$. Assume by contradiction that for $X \in (\epsilon, 2\epsilon)$, $V_2'(X) \geq b((1-A)y_2(X)+\gamma)$, then $V_2'(X)+V_1'(X) > b(1-A)(y_2(X)-y_1(X)) > 0$. Therefore, $V_2'(X) > y_2'(X) > 0$, which implies that $V_2''(X) < 0$, thus for $X \in (\epsilon, 2\epsilon)$, $V_2'(X) < b((1-A)y_2(X) + \gamma)$, a contradiction. In all, for $X \in (\epsilon, 2\epsilon)$, $V_2'(X) < b((1-A)y_2(X) + \gamma)$. By induction, the argument can apply to all $X \in (n\epsilon, (n+1)\epsilon)$ and $n \geq 1$, thus for all $X \in (0, X^*)$. □

Proof of Theorem 3.3:

Proof. **Step 1:** $X > X^*$.

For $X > X^*$, the optimality condition implies that $V_1'(X) \leq 0$ and $V_2'(X) \leq r_2(1-A)$. The value functions satisfy $V_1'(X) = -r_1V_1(X)$ and $V_2'(X) = r_2(1 - V_2(X))$. Therefore, $V_2(X) \geq A$ and $V_1(X) \geq 0$. Specifically, $V_2(X^*) = A$.

Step 2: $0 < X < X^*$.

Consider the optimality condition of firm 1: $V_1'(X) = -r_1(1-A)y_1(X)$, $V_1(X) = y_1(X)$. Therefore,

$$y_1(X) = y_1(0)e^{-r_1(1-A)X},\ y_2(X) = 1 - y_1(0)e^{-r_1(1-A)X}.$$

Since $r_1 > r_2$, $y_1(0) < \frac{1}{2} < y_2(0)$. Furthermore, $a_1(X) = \frac{X}{X^*}$.

Consider the optimality condition of firm 2: $V_2'(X) < r_2(1-A)y_2(X)$ and $r_2V_2(X) = r_2y_2(X) - a_1V_2'(X)$. Therefore,

$$V_2'(X) = \frac{r_2X^*}{X}(1 - y_1(0)e^{-r_1(1-A)X} - V_2(X)).$$

For $0 < X < X^*$, there is a unique solution:

$$V_2(X) = 1 - y_1(0)r_2X^*X^{-r_2X^*}\int_0^X e^{-r_1(1-A)x}x^{r_2X^*-1}dx$$

By optimality condition at X^*, we need $V_2(X^*) \geq A$.

$$h_2(X^*) \equiv r_2y_1(0)(X^*)^{1-r_2X^*}\int_0^{X^*} e^{-r_1(1-A)x}x^{r_2X^*-1}dx \leq 1 - A.$$

Define $\bar{X}_2$ is the solution of $h_2(X) = 0$. Since $h_2(X^*)$ is decreasing in X^*, then the above inequality is equivalent to $X^* \geq \bar{X}_2$.

Similarly, define $h_1(X) = r_1 y_2(0)(X)^{1-r_1 X} \int_0^X e^{-r_2(1-A)x} x^{r_1 X - 1} dx - (1-A)$ and $\bar{X}_1$ is the solution of $h_1(X) = 0$. We need $X^* \geq \bar{X}_1$. We can show that $\bar{X}_1 = \bar{X}_2 \equiv \bar{X}$, which also determines $y_1(0)$ and $y_2(0)$.

Step 3: Check the optimality condition: $V_2'(X) < r_2(1-A)y_2(X)$ for $0 < X < X^*$ and $V_1'(X) > -r_1(1-A)y_1(X)$ for $-X^* < X < 0$.

By similar argument as in Step 3 of Theorem 3.2.

□

Proof of Theorem 3.4:

Proof. Denote $\pi_m = 1$, $\pi_p = 1 - \frac{A}{2}$ and $c = 1 - A + \gamma$.

At state 0, $V_1(0) = V_2(0) = \pi_p - c$. At the state $k \geq 2$, $V_2(k) = \pi_m$ and $V_1(k) = 0$.

Step 1: Check the optimality condition at $k = 1$.

Given the equilibria at $k = 0$ and $k \geq 2$, $V_1(0) = V_2(0) = \pi_p - c$ and $V_2(k) = \pi_m$ and $V_1(k) = 0$ for $k \geq 2$. Denote $a_1 = a_1(1)$ and $a_2 = a_2(1)$. Given $y_1(1) = 0$ and $y_2(1) = 1$, in order for two firms to be indifferent between investing and not investing, then

$$V_2(1) = (1-\delta)(\pi_m - c) + \delta(a_1 V_2(1) + (1-a_1)\pi_m) = (1-\delta)\pi_m + \delta(a_1(\pi_p - c) + (1-a_1)V_2(1)).$$

$$V_1(1) = (1-\delta)(-\gamma) + \delta(a_2 V_1(1) + (1-a_2)(\pi_p - c)) = (1-\delta)0 + \delta(a_2 0 + (1-a_2)V_1(1)).$$

Therefore, $a_2 = 1 - \frac{(1-\delta)\gamma}{\pi_p - c}$ and $V_1(1) = 0$.

Consider the first equality. If $a_1 = 0$ and $LHS \geq RHS$, we can show that $V_2(1) = (1-\delta)(\pi_m - c) + \delta\pi_m$ and $LHS < RHS$, a contradiction. In all, $a_1 = 0$ implies that $LHS < RHS$.

If $a_1 = a_2 = 1 - \frac{(1-\delta)\gamma}{\pi_p - c} \equiv a^*$ and $LHS \leq RHS$, then $V_2(1) = \frac{(1-\delta)\pi_m + \delta a^*(\pi_p - c)}{1-\delta+\delta a^*}$. $LHS \leq RHS$ implies that $(1-\delta)c \geq \frac{1-\delta a^{*2}}{1-\delta+\delta a^*}(\pi_m - \pi_p + c)$. For large δ, this does not hold since $\pi_m > \pi_p$ and $\frac{1-\delta a^{*2}}{1-\delta+\delta a^*} > 1-\delta$, a contradiction. In all, $a_1 = a^*$ implies that $LHS > RHS$.

Therefore, there exist $a_1 \in (0, a_2)$ such that $LHS = RHS$. Therefore, it is optimal for the buyer to choose $y_1(1) = 0$ and $y_2(1) = 1$.

Step 2: Check the optimality condition at $k \geq 2$.

Given $y_1(k) = 0$, $y_2(k) = 1$ and $a_1(k) = 0$, firm 2 has no incentive to invest since it gets the higher payoff 1 and firm 1 also has no incentive to invest since investment leads to state $k-1$ in which $V_1(k-1) = 0$ for $k \geq 2$. Given $a_1(k) = 0$ and $a_2(k) = 0$, the buyer's optimal strategy is $y_1(k) = 0$, $y_2(k) = 1$.

Step 3: Check the optimality condition at $k = 0$.

By symmetry, we can need to check the optimality condition for firm 2. If firm 2 deviates, it gets $(1-\delta)\frac{1}{2} + \delta 0$, which is less than $\pi_p - c$ for large δ.

Step 4: At state $0 < k\Delta < X^*$, it is impossible that $a_2(k) = 1$.

Prove by contradiction. It is true that $y_2(k) = 1$. Firm 1 plays $a_1(k) = 0$ and gets the highest payoff 0 instead of negative payoff. Then, firm 2 has an incentive to deviate to $a_2(k) = 0$, a contradiction.

□

Appendix B. Proofs of Section 3.2

In this section, we prove the results under *one-step up-and-down transition rule.*

Proof of Theorem 3.5:

Proof. **Step 1:** For $0 \leq X_2 - X_1 \leq X^*$, $a_2(X) = 0$, $a_1(X) = \frac{X_2 - X_1}{X^*}$, $y_1(X) \in (0,1)$, $y_2(X) = 1 - y_1(X)$. By symmetry, $y_1(X) = y_2(X) = \frac{1}{2}$ for $X_1 = X_2$.

$$\begin{aligned}
V_2(X_1, X_2) &= (1-\delta)g_2(0, y_2) + \delta\big(a_1 V_2(X_1+\Delta, X_2-\Delta) + (1-a_1)V_2(X_1-\Delta, X_2-\Delta)\big) \\
&> (1-\delta)g_2(1, y_2) + \delta\big(a_1 V_2(X_1+\Delta, X_2+\Delta) + (1-a_1)V_2(X_1-\Delta, X_2+\Delta)\big). \\
V_1(X_1, X_2) &= (1-\delta)g_1(1, y_1) + \delta V_1(X_1+\Delta, X_2-\Delta) \\
&= (1-\delta)g_1(0, y_1) + \delta V_1(X_1-\Delta, X_2-\Delta).
\end{aligned}$$

In the limit $\Delta \to 0$,

$$\begin{aligned}
r(y_2(X) - V_2(X)) &= (1 - 2\frac{X_2 - X_1}{X^*})\frac{\partial V_2(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2}. \\
r\frac{1-A}{1+A}V_1(X) &= \frac{\partial V_1(X)}{\partial X_1} - \frac{1-A}{1+A}\frac{\partial V_1(X)}{\partial X_2}.
\end{aligned}$$

Therefore,

$$\begin{aligned} V_1(X) &= e^{r\frac{1-A}{1+A}X_1} f(X_1 + \frac{1+A}{1-A}X_2). \\ y_1(X) &= \frac{2}{r(1-A)} \frac{\partial V_1(X)}{\partial X_1}. \\ r(1-V_2(X)) &= \frac{2}{1-A}\frac{\partial V_1(X)}{\partial X_1} + (1-2\frac{X_2-X_1}{X^*})\frac{\partial V_2(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2}. \end{aligned}$$

Step 2: Solve $V_1(X_1, X_2)$ and $y_1(X_1, X_2)$ for $0 < X_2 - X_1 \le X^*$ and $X_2 > X^{**}$. Furthermore, $y_1(X_1, X_2)$ only depends on $X_2 - X_1$.

At $X_1 = X_2 = X > X^{**}$, define $v(X) = V_1(X, X) = V_2(X, X)$, then

$$r(\frac{1}{2} - v(X)) = \frac{\partial V_2(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2} = \frac{dv(X)}{dX}.$$

Then, $V_1(X, X) = v(X) = \frac{1}{2} + Ce^{-rX}$. Furthermore, $f(X) = e^{-r\frac{(1-A)^2}{2(1+A)}X}(\frac{1}{2} + Ce^{-\frac{r(1-A)}{2}X})$, $V_1(X_1, X_2) = \frac{1}{2}e^{\frac{r(1-A)}{2}(X_1-X_2)} + Ce^{-rX_2}$ and $y_1(X_1, X_2) = \frac{\partial V_1(X)}{\partial X_1}\frac{2}{r(1-A)} = \frac{1}{2}e^{\frac{r(1-A)}{2}(X_1-X_2)}$. Therefore, $y_1(X_1, X_2)$ only depends on $X_2 - X_1$.

Step 3: Show that $C = 0$, thus $V_1(X_1, X_2)$ and $V_2(X_1, X_2)$ only depends on $X_2 - X_1$.

At $X = (X_1, X_2) = (0, 0)$, $a_1(X) = a_2(X) = 0$ and $y_1(X) = y_2(X) = \frac{1}{2}$ by symmetry. Then, at $X = (X_1, X_2) = (0, 0)$,

$$\begin{aligned} V_2(0,0) &= (1-\delta)g_2(0, \frac{1}{2}) + \delta V_2(0,0). \\ V_1(0,0) &= (1-\delta)g_1(0, \frac{1}{2}) + \delta V_1(0,0). \end{aligned}$$

Therefore, $V_1(0, 0) = V_2(0, 0) = \frac{1}{2}$. Since we have shown that $V_1(X, X) = V_2(X, X) = \frac{1}{2} + Ce^{-bX}$, then $C = 0$. In all, $V_1(X_1, X_2)$ and $V_2(X_1, X_2)$ only depends on $X_2 - X_1$. Define $X = X_2 - X_1$, then

$$\frac{X}{X^*}\frac{\partial V_2(X)}{\partial X} = \frac{r}{2}(1 - \frac{1}{2}e^{\frac{-r(1-A)}{2}X} - V_2(X)).$$

For $0 < X < X^*$, there is a unique solution:

$$V_2(X) = 1 - \frac{rX^*}{4}X^{-\frac{r}{2}X^*}\int_0^X e^{-\frac{r}{2}(1-A)x}x^{\frac{r}{2}X^*-1}dx.$$

Step 4: For $X_2 - X_1 > X^*$, $a_2(X) = 0$, $a_1(X) = 1$, $y_1(X) = 0$ and $y_2(X) = 1$.

$$\begin{aligned} V_2(X_1, X_2) &= (1-\delta)g_2(0,1) + \delta V_2(X_1+\Delta, X_2-\Delta) \geq (1-\delta)g_2(1,1) + \delta V_2(X_1+\Delta, X_2+\Delta) \\ V_1(X_1, X_2) &= (1-\delta)g_1(1,0) + \delta V_1(X_1+\Delta, X_2-\Delta) \geq (1-\delta)g_1(0,0) + \delta V_1(X_1-\Delta, X_2-\Delta). \end{aligned}$$

In the limit $\Delta \to 0$, $r(1 - V_2(X)) = \frac{\partial V_2(X)}{\partial X_2} - \frac{\partial V_2(X)}{\partial X_1}$ and $rV_1(X) = \frac{\partial V_1(X)}{\partial X_1} - \frac{\partial V_1(X)}{\partial X_2}$. Since $V_1(X_1, X_2)$ and $V_1(X_1, X_2)$ only depends on $X \equiv X_2 - X_1$, then $\frac{\partial V_2(X)}{\partial X} = \frac{r}{2}(1 - V_2(X)) \geq \frac{r(1-A)}{2}$ and $\frac{\partial V_1(X)}{\partial X} = \frac{r}{2}V_1(X)$. Therefore, $X \equiv X_2 - X_1 > X^*$

$$\begin{aligned} V_1(X) &= e^{\frac{r}{2}X}V_1(X^*) = \frac{1}{2}e^{\frac{r}{2}X}e^{-\frac{r(1-A)}{2}X^*}. \\ V_2(X) &= 1 - e^{-\frac{r}{2}X}V_2(X^*). \end{aligned}$$

Step 5: Check the optimality condition: $V_2(X^*) \geq A$ and $V_2'(X) < \frac{r(1-A)}{2}y_2(X)$ for $0 < X < X^*$.

In order to show $V_2(X^*) \geq A$, we need

$$h(X^*) \equiv \frac{rX^*}{4}(X^*)^{-\frac{r}{2}X^*} \int_0^{X^*} e^{-\frac{r}{2}(1-A)x} x^{\frac{r}{2}X^*-1} dx \leq 1 - A.$$

Define $\tilde{X}$ as the solution of $h(X^*) = 0$. Since $h(X^*)$ is decreasing in X^*, then the above inequality is equivalent to $X^* \geq \tilde{X}$.

Take derivative of $\frac{r}{2}V_2(X) = \frac{r}{2}y_2(X) - \frac{X}{X^*}V_2'(X)$, we have $(\frac{r}{2} + \frac{1}{X^*})V_2'(X) + \frac{X}{X^*}V_2''(X) = \frac{r}{2}y_2'(X)$. At $X = 0$, $V_2'(0) = \frac{\frac{1}{8}r^2(1-A)}{\frac{r}{2}+\frac{1}{X^*}} < r\frac{1-A}{4} = r\frac{1-A}{2}y_2(0)$, which satisfy the optimality condition. Furthermore, $V_2''(0) = -(\frac{r}{2})^3(1 - A^2)\frac{1}{2}(\frac{r}{2} + \frac{2}{X^*})^{-1} < 0$. Therefore, $V_2'(X) < \frac{r(1-A)}{2}y_2(X)$ at $X \in (0, \epsilon)$. Assume by contradiction that for $X \in (\epsilon, 2\epsilon)$, $V_2'(X) \geq \frac{r(1-A)}{2}y_2(X)$, then $V_2'(X)+V_1'(X) > \frac{r(1-A)}{2}(y_2(X)-y_1(X)) > 0$. Therefore, $V_2'(X) > y_2'(X) > 0$, which implies that $V_2''(X) < 0$, thus for $X \in (\epsilon, 2\epsilon)$, $V_2'(X) < \frac{r(1-A)}{2}y_2(X)$, a contradiction. In all, for $X \in (\epsilon, 2\epsilon)$, $V_2'(X) < \frac{r(1-A)}{2}y_2(X)$. By induction, the argument can apply to all $X \in (n\epsilon, (n+1)\epsilon)$ and $n \geq 1$, thus for all $X \in (0, X^*)$.

□

Proof of Theorem 3.6:

Proof. For simplicity, assume that $\gamma = 0$. Denote $\pi_m = 1$, $\pi_p = 1 - \frac{A}{2}$ and $c = 1 - A + \gamma$.

$V_i(X_1, X_2)$ only depends on $X_2 - X_1$. At state $(0,0)$, $V_1(0,0) = V_2(0,0) = \pi_p - c$, $y_1(0,0) = y_2(0,0) = \frac{1}{2}$, $a_1(0,0) = a_2(0,0) = 1$. Define $k \equiv X_2 - X_1$. At the state $k \geq 1$, $V_1(k) = 0$,

$y_2(k) = 1$, $y_1(k) = 0$, $y_2(k) = 1$. At the state $k \geq 3$, $V_2(k) = \pi_m$, $V_1(k) = 0$, $y_2(k) = 1$, $y_1(k) = 0$, $a_1(k) = a_2(k) = 0$.

Step 1: Check the optimality condition for $X_1 = 0$.

(1) At state $(0, 1)$, denote $a_1 = a_1(0, 1)$ and $a_2 = a_2(0, 1)$.

$$V_2(0,1) = (1-\delta)(\pi_m-c)+\delta(a_1V_2(1,2)+(1-a_1)V_2(0,2)) > (1-\delta)\pi_m+\delta(a_1V_2(1,0)+(1-a_1)V_2(0,0)).$$

$$V_1(0,1) = (1-\delta)0+\delta(a_2V_1(0,2)+(1-a_2)V_1(0,0)) = (1-\delta)0+\delta(a_2V_1(1,2)+(1-a_2)V_1(1,0)).$$

Therefore, $V_1(0, 1) = V_1(0, 2) = V_1(1, 2) = 0$ and $a_2(0, 1) = 1$. If $a_1(0, 1) = 1$, then $LHS = \pi_m - c > RHS = (1 - \delta)\pi_m$. In all, $a_1(0, 1) = a_2(0, 1) = 1$.

(2) At state $(0, 2)$, denote $a_1 = a_1(0, 2)$ and $a_2 = a_2(0, 2)$.

$$V_2(0,2) = (1-\delta)(\pi_m-c)+\delta(a_1V_2(1,3)+(1-a_1)V_2(0,3)) > (1-\delta)\pi_m+\delta(a_1V_2(1,1)+(1-a_1)V_2(0,1)).$$

$$V_1(0,2) = (1-\delta)0+\delta(a_2V_1(0,3)+(1-a_2)V_1(0,1)) = (1-\delta)0+\delta(a_2V_1(1,3)+(1-a_2)V_1(1,1)).$$

Since $V_1(0, 2) = V_1(0, 3) = V_1(0, 1) = V_1(1, 3) = 0$, then $a_2 = 1$. If $a_1 = 1$, then $LHS = \pi_m - c > RHS = (1 - \delta)\pi_m + \delta(\pi_p - c)$ for large δ. In all, $a_1(0, 2) = a_2(0, 2) = 1$.

(3) At state $(0, k)$ where $k \geq 3$, $a_1(0, k) = 0$ and $a_2(0, k) = 0$.

$$V_1(0, k) = (1 - \delta)0 + \delta V_1(0, k - 1) \geq (1 - \delta)0 + \delta V_1(1, k - 1).$$

Step 2: Check the optimality condition at $k = 1$ and $X_1 \geq 1$.

Given the equilibria at $k = 0$ and $k \geq 2$, $V_1(0) = V_2(0) = \pi_p - c$ and $V_2(k) = \pi_m$ and $V_1(k) = 0$ for $k \geq 2$. Denote $a_1 = a_1(1)$ and $a_2 = a_2(1)$. Given $y_1(1) = 0$ and $y_2(1) = 1$, in order for two firms to be indifferent between investing and not investing, then

$$V_2(1) = (1-\delta)(\pi_m-c)+\delta(a_1V_2(1)+(1-a_1)V_2(2)) = (1-\delta)\pi_m+\delta(a_1V_1(1)+(1-a_1)V_2(1)).$$

$$V_1(1) = (1 - \delta)0 + \delta(a_2V_1(1) + (1 - a_2)V_2(1)) = (1 - \delta)0 + \delta(a_2V_1(3) + (1 - a_2)V_1(1)).$$

Therefore, $V_1(1) = 0$ and $a_2(1) = 1$. Consider the first equality. If $a_1 = 0$, then $LHS = (1 - \delta)(\pi_m - c) + \delta V_2(2) < (1 - \delta)\pi_m + \delta V_2(1) = RHS$. This is equivalent to $(1 - \delta)c > \delta(V_2(2) - V_2(1)) = \delta(1 - \delta)(V_2(2) - \pi_m + c)$, which is true since $V_2(2) < \pi_m$. If $a_1 = 1$, then $LHS = \pi_m - c > (1 - \delta)\pi_m = RHS$ for large δ. Therefore, there exist $a_1(1) \in (0, 1)$ and it is optimal for the buyer to choose $y_1(1) = 0$ and $y_2(1) = 1$.

Step 3: Check the optimality condition at $k = 2$ and $X_1 \geq 1$.

Denote $a_1 = a_1(2)$ and $a_2 = a_2(2)$. Given $y_1(1) = 0$ and $y_2(1) = 1$, in order for two firms to be indifferent between investing and not investing, then

$$V_2(2) = (1-\delta)(\pi_m - c) + \delta(a_1 V_2(2) + (1-a_1)V_2(3)) = (1-\delta)\pi_m + \delta(a_1 V_2(0) + (1-a_1)V_2(2)).$$

$$V_1(2) = (1-\delta)0 + \delta(a_2 V_1(2) + (1-a_2)V_2(0)) = (1-\delta)0 + \delta(a_2 V_1(4) + (1-a_2)V_1(2)).$$

Therefore, $V_1(2) = 0$ and $a_2(2) = 1$. Consider the first equality. If $a_1 = 0$ and $LHS \geq RHS$, then $V_2(2) = (1-\delta)(\pi_m - c) + \delta\pi_m$ and $LHS = (1-\delta)(\pi_m - c) + \delta\pi_m < (1-\delta)\pi_m + \delta V_2(2) = RHS$, a contradiction. Therefore, $a_1 = 0$ implies $LHS < RHS$.

If $a_1 = 1$, then $LHS > RHS$ for large δ. Therefore, there exist $a_1(2) \in (0,1)$ to satisfy the indifference condition for firm 2 and it is optimal for the buyer to choose $y_1(2) = 0$ and $y_2(2) = 1$.

Step 4: Check the optimality condition at $k \geq 3$.

Given $y_1(k) = 0$, $y_2(k) = 1$ and $a_1(k) = 0$, firm 2 has no incentive to invest since it gets the higher payoff 1 and firm 1 also has no incentive to invest since investment leads to state $k-1$ in which $V_1(k-2) = 0$ for $k \geq 3$. Given $a_1(k) = 0$ and $a_2(k) = 0$, the buyer's optimal strategy is $y_1(k) = 0$, $y_2(k) = 1$.

Step 5: Check the optimality condition at $X_1 = X_2$.

By symmetry, we can need to check the optimality condition for firm 2. If firm 2 deviates, it gets $(1-\delta)\frac{1}{2} + \delta 0$, which is less than $\pi_p - c$ for large δ.

□

Appendix C. Proofs of Section 4

Proof of Theorem 4.1:

Proof. For notational simplicity, let $g_1(0,1) = 1$ and $g_1(1,1) = A \in (0,1)$.

Step 1: Characterize the equilibrium for $X \in [0, X^{**}] \times [0, X^{**}]$,

If $X \in [0, X^{**}] \times [0, X^{**}]$, then all parties play mixed strategies in the unique stationary Markov equilibrium describe below. By symmetry, we only need to check the optimal condition of firm 2: $V_2(X_1, X_2) = (1-\delta)g_2(1, y_2) + \delta\big(a_1 V_2(X_1+\Delta, X_2+\Delta) + (1-a_1)V_2(X_1, X_2+\Delta)\big)$ In the limit as $\Delta \to 0$, the above equation is equivalent to $rV_2(X) = ry_2(X) + a_1(X)\frac{\partial V_2(X)}{\partial X_1}$

and $\frac{\partial V_2(X)}{\partial X_2} = r(1-A)y_2(X)$. Then, for $-X^* < X < X^*$,

$$\frac{\partial V_2(X)}{\partial X_2} + a_1(X)(1-A)\frac{\partial V_2(X)}{\partial X_1} = r(1-A)V_2(X). \tag{C.1}$$

Similarly, for $-X^* < X < X^*$,

$$\frac{\partial V_1(X)}{\partial X_1} + a_2(X)(1-A)\frac{\partial V_1(X)}{\partial X_2} = r(1-A)V_1(X). \tag{C.2}$$

Furthermore, the buyer needs to indifferent between B_1, B_2 and NB. Therefore,

$$a_i(X) = \frac{-\alpha_0}{\alpha_1-\alpha_0} - \frac{\beta X_i}{\alpha_1-\alpha_0} = \frac{X^{**}-X_i}{X^*}.$$

By solving (1) and (2), we have

$$V_1(X) = e^{r(1-A)X_1}\phi(e^{\frac{1-A}{X^*}X_1}\frac{X^{**}-X_2}{X^*}).$$
$$V_2(X) = e^{r(1-A)X_2}\phi(e^{\frac{1-A}{X^*}X_2}\frac{X^{**}-X_1}{X^*}).$$

$\phi(\cdot)$ is an arbitrary function.

Under the assumption that $X^* > \bar{X}$, by Theorem 3.1, we know that $V_1(X_1, X^{**}) = \frac{1}{2}e^{r(1-A)(X_1-X^{**})}$ and

$$V_2(X_1, X^{**}) = 1 - \frac{rX^*}{2}(X^{**}-X_1)^{-rX^*}\int_0^{X^{**}-X_1} e^{-r(1-A)x}x^{rX^*-1}dx.$$

Value matching $V_1(X_1^-, X^{**}) = V_1(X_1^+, X^{**})$ implies that $\phi(0) = \frac{1}{2}e^{-b(1-A)X^{**}}$. Value matching $V_2(X_1^-, X^{**}) = V_2(X_1^+, X^{**})$ implies that

$$e^{r(1-A)X^{**}}\phi(e^{\frac{1-A}{X^*}X^{**}}\frac{X^{**}-X_1}{X^*}) = 1 - \frac{rX^*}{2}(X^{**}-X_1)^{-rX^*}\int_0^{X^{**}-X_1} e^{-r(1-A)x}x^{bX^*-1}dx.$$

Therefore,

$$\phi(X) = e^{-r(1-A)X^{**}}(1 - \frac{rX^*}{2}(MX)^{-rX^*}\int_0^{MX} e^{-r(1-A)x}x^{rX^*-1}dx),$$

where $M = X^* e^{-(1-A)\frac{X^{**}}{X^*}}$. In all,

$$V_1(X) = e^{r(1-A)(X_1-X^{**})} - \frac{rX^*}{2}(X^{**}-X_2)^{-rX^*}\int_0^{e^{\frac{1-A}{X^*}(X_1-X^{**})}(X^{**}-X_2)} e^{-r(1-A)x}x^{rX^*-1}dx.$$

$$V_2(X) = e^{r(1-A)(X_2-X^{**})} - \frac{rX^*}{2}(X^{**}-X_1)^{-rX^*}\int_0^{e^{\frac{1-A}{X^*}(X_2-X^{**})}(X^{**}-X_1)} e^{-r(1-A)x}x^{rX^*-1}dx.$$

Step 2: Show that $y_1(X_1, X^{**}+) = y_1(X_1, X^{**}-)$, $y_2(X_1, X^{**}+) = y_2(X_1, X^{**}-)$.

For $X_2 < X^{**}$, $y_1(X) = \frac{1}{r(1-A)}\frac{\partial V_1(X)}{\partial X_1} = [1+\frac{1-A}{X^*}(X^{**}-X_2)]V_1(X)$. Therefore, $y_1(X_1, X^{**}-) = \frac{1}{2}e^{r(1-A)(X_1-X^{**})}$. For $X_2 > X^{**}$, $y_1(X_1, X_2) = \frac{1}{2}e^{r(1-A)(X_1-X_2)}$, then $y_1(X_1, X^{**}+) = \frac{1}{2}e^{r(1-A)(X_1-X^{**})}$. In all, $y_1(X_1, X^{**}-) = y_1(X_1, X^{**}+)$.

If $X_2 = X^{**}$, then $rV_2(X) - a_1(X)\frac{\partial V_2(X)}{\partial X_1} = rV_2(X) + a_1(X)\frac{\partial V_2(X)}{\partial(X^{**}-X_1)} = r(1-\frac{1}{2}e^{r(1-A)(X_1-X^{**})})$. If $X \in [0, X^{**}] \times [0, X^{**}]$, $\frac{\partial V_2(X)}{\partial X_2} + a_1(X)(1-A)\frac{\partial V_2(X)}{\partial X_1} = r(1-A)V_2(X)$. Therefore, the value matching of $\frac{\partial V_2(X)}{\partial X_1}$ implies that

$$y_2(X_1, X^{**}+) = 1 - \frac{1}{2}e^{r(1-A)(X_1-X^{**})} = \frac{1}{r(1-A)}\frac{\partial V_2(X)}{\partial X_2} = y_2(X_1, X^{**}-).$$

Step 3: $\frac{\partial y_1(X)}{\partial X_1} > 0$ and $\frac{\partial y_1(X)}{\partial X_2} < 0$.

Since $y_1(X) = [1 + \frac{1-A}{X^*}(X^{**} - X_2)]V_1(X)$, $\frac{\partial y_1(X)}{\partial X_1} = [1 + \frac{1-A}{X^*}(X^{**} - X_2)]\frac{\partial V_1(X)}{\partial X_1} > 0$. Then, $\frac{\partial y_1(X)}{\partial X_2} = [1+(1-A)a_2(X)]\frac{\partial V_1(X)}{\partial X_2} - \frac{1-A}{X^*}V_1(X)$. By $\frac{\partial V_1(X)}{\partial X_1} + a_2(X)(1-A)\frac{\partial V_1(X)}{\partial X_2} = r(1-A)V_1(X)$ and $\frac{\partial V_1(X)}{\partial X_1} = r(1-A)y_1(X) = r(1-A)(1+(1-A)a_2(X))V_1(X)$, we have $\frac{\partial V_1(X)}{\partial X_2} = -r(1-A)V_1(X) < 0$. Therefore, $\frac{\partial y_1(X)}{\partial X_2} < 0$.

Step 4: $\frac{\partial V_1(X)}{\partial X_1} > -\frac{\partial V_1(X)}{\partial X_2}$. The sign of $\frac{\partial y_1(X)}{\partial X_1} + \frac{\partial y_1(X)}{\partial X_2}$ is not determined.

$$\frac{\partial V_1(X)}{\partial X_1} + \frac{\partial V_1(X)}{\partial X_2} = r(1-A)^2 a_2(X)V_1(X) > 0.$$

$$\frac{\partial y_1(X)}{\partial X_1} + \frac{\partial y_1(X)}{\partial X_2} = \frac{1-A}{X^*}V_1(X)[rX^*(1-A)a_2(X)(1+(1-A)a_2(X)) - 1].$$

If $X_2 \to X^{**}$, then $a_2(X) \to 0$, thus $\frac{\partial y_1(X)}{\partial X_1} + \frac{\partial y_1(X)}{\partial X_2} < 0$, which means that firm i's market share depends more on firm j's reputation as long as firm j's reputation can beat the buyer's outside option.

□

Proof of Theorem 4.2:

Proof. Consider the case $X \in [0, X^{**}] \times [0, X^{**}]$.

Denote $k = X_2 - X_1$, $\pi_m = 1$, $\pi_p = 1 - \frac{A}{2}$ and $c = 1 - A + \gamma$.

Step 1: Check at the optimality condition if $X_2 - X_1 \geq 2$.

Construct an equilibrium where $a_2(X) = \frac{X^{**}-X_2}{X^*}$, $a_1(X) = 0$, $y_1(X) = 0$, $V_1(X) = 0$. Moreover, $V_2(X)$ only depends on $X_2 - X_1$. Denote V_k as the continuation value of the leading firm 2. Denote $K = \frac{X^{**}}{\Delta} + 1$. For $2 \leq k \leq K-1$, $V_k = (1-\delta)y_k + \delta V_k = (1-\delta)(Ay_k - \gamma) + \delta V_{k+1}$. Then, $y_{k+1} + \frac{\gamma}{1-A} = \frac{1-(1-\delta)A}{\delta}(y_k + \frac{\gamma}{1-A})$. At $k = K$, $V_K = 1$, $a_2(K) = 0$ and $y_2(k) = 1$

by the equilibrium for $X \notin M$. Therefore, $y_k + \frac{\gamma}{1-A} = (\frac{1-(1-\delta)A}{\delta})^{k-K}(1+\frac{\gamma}{1-A})$. Denote $y(X) = \lim_{\Delta\to 0,\ k\Delta\to X} y_{k\Delta}$ and take limit $\Delta \to 0$, $y(X) + \frac{\gamma}{1-A} = e^{r(1-A)(X-X^{**})}(1+\frac{\gamma}{1-A})$.

For $k \geq 2$, firm 1 also has no incentive to invest since investing leads to state $k-1$ with positive probability in which $V_1(k-1) = 0$ for $k \geq 2$.

Given $a_1(k) = 0 < a_2(k) = a_2(X) = \frac{X^{**}-X_2}{X^*}$, the buyer's optimal strategy is $y_1(k) = 0$, $y_2(k) \in (0,1)$.

Step 2: Check at the optimality condition of both firms if $0 \leq k \leq 1$.

Given the equilibria at $k = 0$ and $k \geq 2$, $V_1(0) = V_2(0) = \pi_p - c$ and $V_2(k) = \pi_m$ and $V_1(k) = 0$ for $k \geq 2$. Denote $a_1 = a_1(1)$ and $a_2 = a_2(1)$. Given $y_1(1) = 0$ and $y_2(1) = 1$, in order for two firms to be indifferent between investing and not investing, then

$$V_2(1) = (1-\delta)(\pi_m - c) + \delta(a_1 V_2(1) + (1-a_1)V_2(2)) = (1-\delta)\pi_m + \delta(a_1(\pi_p - c) + (1-a_1)V_2(1)).$$

$$V_1(1) = (1-\delta)(-\gamma) + \delta(a_2 V_1(1) + (1-a_2)(\pi_p - c)) = (1-\delta)0 + \delta(a_2 0 + (1-a_2)V_1(1)).$$

Therefore, $a_2 = 1 - \frac{(1-\delta)\gamma}{\pi_p - c}$ and $V_1(1) = 0$.

Consider the first equality. If $a_1 = 0$ and $LHS \geq RHS$, we can show that $V_2(1) = (1-\delta)(\pi_m - c) + \delta V_2(2)$ and $LHS < RHS$, a contradiction. In all, $a_1 = 0$ implies that $LHS < RHS$.

If $a_1 = a_2 = 1 - \frac{(1-\delta)\gamma}{\pi_p - c} \equiv a^*$ and $LHS \leq RHS$, then $V_2(1) = \frac{(1-\delta)\pi_m + \delta a^*(\pi_p - c)}{1-\delta+\delta a^*}$. $LHS \leq RHS$ implies that $(1-\delta)c \geq \frac{1-\delta a^{*2}}{1-\delta+\delta a^*}(\pi_m - \pi_p + c)$. For large δ, this does not hold since $\pi_m > \pi_p$ and $\frac{1-\delta a^{*2}}{1-\delta+\delta a^*} > 1-\delta$, a contradiction. In all, $a_1 = a^*$ implies that $LHS > RHS$.

There exists $a_1 \in (0, a_2)$ such that $LHS = RHS$. $a_1 < a_2$ implies that $y_1(1) = 0$ and $y_2(1) = 1$.

□

Proof of Theorem 4.3

Proof. **Case 1:** $X \notin [0, X^{**}] \times [0, X^{**}]$ and $X_2 \geq X_1$.

Step 1: For $0 \le X_2 - X_1 \le X^*$, $a_2(X) = 0$, $a_1(X) = \frac{X_2 - X_1}{X^*}$, $y_1(X) \in (0,1)$, $y_2(X) = 1 - y_1(X)$. By symmetry, $y_1(X) = y_2(X) = \frac{1}{2}$ for $X_1 = X_2$.

$$\begin{aligned}
V_2(X_1, X_2) &= (1-\delta)g_2(0, y_2) + \delta\big(a_1 V_2(X_1+\Delta, X_2-\Delta) + (1-a_1)V_2(X_1-\Delta, X_2-\Delta)\big) \\
&> (1-\delta)g_2(1, y_2) + \delta\big(a_1 V_2(X_1+\Delta, X_2+\Delta) + (1-a_1)V_2(X_1-\Delta, X_2+\Delta)\big). \\
V_1(X_1, X_2) &= (1-\delta)g_1(1, y_1) + \delta V_1(X_1+\Delta, X_2-\Delta) \\
&= (1-\delta)g_1(0, y_1) + \delta V_1(X_1-\Delta, X_2-\Delta).
\end{aligned}$$

In the limit $\Delta \to 0$,

$$\begin{aligned}
r(y_2(X) - V_2(X)) &= (1 - 2\frac{X_2 - X_1}{X^*})\frac{\partial V_2(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2}. \\
r\frac{1-A}{1+A}V_1(X) &= \frac{\partial V_1(X)}{\partial X_1} - \frac{1-A}{1+A}\frac{\partial V_1(X)}{\partial X_2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_1(X) &= e^{r\frac{1-A}{1+A}X_1} f(X_1 + \frac{1+A}{1-A}X_2). \\
y_1(X) &= \frac{2}{r(1-A)}\frac{\partial V_1(X)}{\partial X_1}. \\
r(1 - V_2(X)) &= \frac{2}{1-A}\frac{\partial V_1(X)}{\partial X_1} + (1 - 2\frac{X_2 - X_1}{X^*})\frac{\partial V_2(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2}.
\end{aligned}$$

Step 2: Solve $V_1(X_1, X_2)$ and $y_1(X_1, X_2)$ for $0 < X_2 - X_1 \le X^*$ and $X_2 > X^{**}$. Furthermore, $y_1(X_1, X_2)$ only depends on $X_2 - X_1$.

At $X_1 = X_2 = X > X^{**}$, define $v(X) \equiv V_1(X, X) = V_2(X, X)$, then

$$r(\frac{1}{2} - v(X)) = \frac{\partial V_2(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2} = \frac{dv(X)}{dX}.$$

Then, $V_1(X, X) = v(X) = \frac{1}{2} + Ce^{-rX}$, where $C < 0$. Furthermore, $f(X) = e^{-r\frac{(1-A)^2}{2(1+A)}X}(\frac{1}{2} + Ce^{-\frac{r(1-A)}{2}X})$, $V_1(X_1, X_2) = \frac{1}{2}e^{\frac{r(1-A)}{2}(X_1 - X_2)} + Ce^{-rX_2}$ and $y_1(X_1, X_2) = \frac{\partial V_1(X)}{\partial X_1}\frac{2}{r(1-A)} = \frac{1}{2}e^{\frac{r(1-A)}{2}(X_1 - X_2)}$. Therefore, $y_1(X_1, X_2)$ only depends on $X_2 - X_1$.

Step 3: Check the optimality condition $\frac{\partial V_2(X)}{\partial X_2} \le \frac{r(1-A)}{2}y_2(X)$ for $0 \le X_2 - X_1 \le X^*$.

By $\frac{\partial V_1(X)}{\partial X_1} \le \frac{r(1-A)}{2}y_1(X)$ and $y_1(X) + y_2(X) = 1$, $\frac{\partial V_2(X)}{\partial X_2} \le \frac{r(1-A)}{2}y_2(X)$ is equivalent to $\frac{\partial V_2(X)}{\partial X_2} + \frac{\partial V_1(X)}{\partial X_1} < \frac{r(1-A)}{2}$.

Check the optimality condtion at $X_1 = X_2 \geq X^{**}$: $\frac{\partial V_2(X)}{\partial X_2} < \frac{r(1-A)}{4}$. At $X_1 = X_2$, $y_1 = y_2 = \frac{1}{2}$, $V_1 = V_2$. Therefore,

$$\begin{aligned} r(y_2(X) - V_2(X)) &= (1 - 2\frac{X_2 - X_1}{X^*})\frac{\partial V_2(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2}. \\ r(\frac{\partial y_2(X)}{\partial X_2} - \frac{\partial V_2(X)}{\partial X_2}) &= \frac{\partial^2 V_2(X)}{\partial X_1 \partial X_2} + \frac{\partial^2 V_2(X)}{\partial^2 X_2} - \frac{2}{X^*}\frac{\partial V_2(X)}{\partial X_1} - 2\frac{X_2 - X_1}{X^*}\frac{\partial^2 V_2(X)}{\partial X_1 \partial X_2}. \end{aligned}$$

Show that $\frac{\partial V_1(X)}{\partial X_1} = \frac{\partial y_1(X)}{\partial X_1}$ at $X_1 = X_2$. We know that $r\frac{1-A}{1+A}V_1(X) = \frac{\partial V_1(X)}{\partial X_1} - \frac{1-A}{1+A}\frac{\partial V_1(X)}{\partial X_2}$. Therefore, $r\frac{1-A}{1+A}\frac{\partial V_1(X)}{\partial X_1} = \frac{\partial^2 V_1(X)}{\partial^2 X_1} - \frac{1-A}{1+A}\frac{\partial^2 V_1(X)}{\partial X_2 \partial X_1}$. By $\frac{\partial y_1(X)}{\partial X_1} + \frac{\partial y_1(X)}{\partial X_2} = 0$ for $X_1 = X_2$, $\frac{\partial^2 V_1(X)}{\partial^2 X_1} + \frac{\partial^2 V_1(X)}{\partial X_2 \partial X_1} = 0$. Then, $\frac{\partial^2 V_1(X)}{\partial^2 X_1} = \frac{r(1-A)}{2}\frac{\partial V_1(X)}{\partial X_1}$, thus $\frac{\partial V_1(X)}{\partial X_1} = \frac{\partial y_1(X)}{\partial X_1}$.

Assume that $\frac{\partial V_2(X)}{\partial X_2} > \frac{r(1-A)}{4} = \frac{\partial V_1(X)}{\partial X_1}$, then $\frac{\partial y_2(X)}{\partial X_2} = \frac{\partial y_1(X)}{\partial X_1}$ implies that $\frac{\partial V_2(X)}{\partial X_2} > \frac{\partial y_2(X)}{\partial X_2}$. Therefore, $\frac{\partial^2 V_2(X)}{\partial X_1 \partial X_2} + \frac{\partial^2 V_2(X)}{\partial^2 X_2} - \frac{2}{X^*}\frac{\partial V_2(X)}{\partial X_1} < 0$. Then, $\frac{2}{X^*}\frac{\partial V_2(X)}{\partial X_1} < 0$ implies that

$$\frac{\partial^2 V_2(X)}{\partial X_1 \partial X_2} + \frac{\partial^2 V_2(X)}{\partial^2 X_2} < 0,$$

which means that in the direction $X_1 = X_2$, $d(\frac{\partial V_2(X)}{\partial X_2}) = (\frac{\partial V_2(X)}{\partial X_1 \partial X_2} + \frac{\partial^2 V_2(X)}{\partial^2 X_2})d(X_1) < 0$, a contradiction to $\frac{\partial V_2(X)}{\partial X_2} = \frac{r(1-A)}{4}$ at $X_1 = X_2 = X^{**}$ and $\frac{\partial V_2(X)}{\partial X_2} > \frac{r(1-A)}{4}$ for any $X_1 = X_2 > X^{**}$.

Step 4: For $X_2 - X_1 > X^*$, $a_2(X) = 0$, $a_1(X) = 1$, $y_1(X) = 0$ and $y_2(X) = 1$.

$$\begin{aligned} V_2(X_1, X_2) &= (1-\delta)g_2(0,1) + \delta V_2(X_1 + \Delta, X_2 - \Delta) \geq (1-\delta)g_2(1,1) + \delta V_2(X_1 + \Delta, X_2 + \Delta) \\ V_1(X_1, X_2) &= (1-\delta)g_1(1,0) + \delta V_1(X_1 + \Delta, X_2 - \Delta) \geq (1-\delta)g_1(0,0) + \delta V_1(X_1 - \Delta, X_2 - \Delta). \end{aligned}$$

In the limit $\Delta \to 0$,

$$\begin{aligned} r(1 - V_2(X)) &= \frac{\partial V_2(X)}{\partial X_2} - \frac{\partial V_2(X)}{\partial X_1}. \\ rV_1(X) &= \frac{\partial V_1(X)}{\partial X_1} - \frac{\partial V_1(X)}{\partial X_2}. \end{aligned}$$

We also need $\frac{\partial V_2(X)}{\partial X_2} \leq \frac{r(1-A)}{2}y_2(X)$ and $\frac{\partial V_1(X)}{\partial X_1} > 0$. Therefore,

$$\begin{aligned} V_1(X) &= e^{rX_1}g_1(X_1 + X_2). \\ V_2(X) &= 1 - e^{-rX_2}g_2(X_1 + X_2). \end{aligned}$$

Value Matching at $X_2 - X_2 = X^*$ implies that

$$e^{rX_1}g_1(2X_1 + X^*) = Ce^{-b(X^* + X_1)} + \frac{1}{2}e^{\frac{-r(1-A)}{2}X^*}.$$

$$g_1(Y) = Ce^{-rY} + \frac{1}{2}e^{\frac{rA}{2}X^*}e^{-\frac{rY}{2}}.$$

Therefore, for $X_2 - X_1 \geq X^*$,

$$V_1(X) = Ce^{-rX_2} + \frac{1}{2}e^{\frac{rA}{2}X^*}e^{\frac{r}{2}(X_1-X_2)}.$$

Case 2: $X \in [0, X^{**}] \times [0, X^{**}]$ and $X_2 \geq X_1$.

Step 1: Study the equilibrium if the buyer mixes between B_1, B_2 and NB.

$$\begin{aligned}
V_2(X_1, X_2) &= (1-\delta)g_2(1, y_2) + \delta\big(a_1V_2(X_1+\Delta, X_2+\Delta) + (1-a_1)V_2(X_1-\Delta, X_2+\Delta)\big) \\
&= (1-\delta)g_2(0, y_2) + \delta\big(a_1V_2(X_1+\Delta, X_2-\Delta) + (1-a_1)V_2(X_1-\Delta, X_2-\Delta)\big). \\
V_1(X_1, X_2) &= (1-\delta)g_1(1, y_1) + \delta\big(a_2V_1(X_1+\Delta, X_2+\Delta) + (1-a_2)V_1(X_1+\Delta, X_2-\Delta)\big) \\
&= (1-\delta)g_1(0, y_1) + \delta\big(a_2V_1(X_1-\Delta, X_2+\Delta) + (1-a_2)V_1(X_1-\Delta, X_2-\Delta)\big).
\end{aligned}$$

In the limit as $\Delta \to 0$,

$$\begin{aligned}
rV_2(X) &= ry_2(X) - (1-2a_1(X))\frac{\partial V_2(X)}{\partial X_1} - \frac{\partial V_2(X)}{\partial X_2}. \\
\frac{\partial V_2(X)}{\partial X_2} &= \frac{r}{2}(1-A)y_2(X).
\end{aligned}$$

Similarly,

$$\begin{aligned}
rV_1(X) &= ry_1(X) - (1-2a_2(X))\frac{\partial V_1(X)}{\partial X_2} - \frac{\partial V_1(X)}{\partial X_1}. \\
\frac{\partial V_1(X)}{\partial X_1} &= \frac{r}{2}(1-A)y_1(X).
\end{aligned}$$

The optimality condtion for $X_1 \in (0, X^{**})$ and $X_2 \in (0, X^{**})$ is

$$\begin{aligned}
\frac{\partial V_1(X)}{\partial X_1} - (1-2a_2(X))\frac{1-A}{1+A}\frac{\partial V_1(X)}{\partial X_2} &= r\frac{1-A}{1+A}V_1(X). \\
\frac{\partial V_2(X)}{\partial X_2} - (1-2a_1(X))\frac{1-A}{1+A}\frac{\partial V_2(X)}{\partial X_1} &= r\frac{1-A}{1+A}V_2(X).
\end{aligned}$$

Therefore, we can solve for $V_1(X)$ and $V_2(X)$.

$$\begin{aligned}
V_1(X) = e^{r\frac{1-A}{1+A}X_1}\phi(e^{\frac{1-A}{1+A}\frac{2X_1}{X^*}}(1 - 2\frac{X^{**}-X_2}{X^*})). \\
V_2(X) = e^{r\frac{1-A}{1+A}X_2}\phi(e^{\frac{1-A}{1+A}\frac{2X_2}{X^*}}(1 - 2\frac{X^{**}-X_1}{X^*})).
\end{aligned}$$

Step 2: There exists an increasing function $f(X_1, X_2)$ such that the buyer mixes between B_1, B_2 and NB if $f(X_1, X_2) \leq 0$ and mix between B_1 and B_2 if $f(X_1, X_2) \geq 0$.

If $f(X_1, X_2) \geq 0$, the buyer mixes between $B1$, $B2$ and the firm also play mixed strategy as follows:

$$\begin{aligned}
\frac{\partial V_1(X)}{\partial X_1} - (1-2a_2(X))\frac{1-A}{1+A}\frac{\partial V_1(X)}{\partial X_2} &= r\frac{1-A}{1+A}V_1(X). \\
\frac{\partial V_2(X)}{\partial X_2} - (1-2a_1(X))\frac{1-A}{1+A}\frac{\partial V_2(X)}{\partial X_1} &= r\frac{1-A}{1+A}V_2(X). \\
\frac{\partial V_1(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2} &= \frac{r(1-A)}{2}. \\
a_1(X) - a_2(X) &= \frac{X_2 - X_1}{X^*}.
\end{aligned}$$

We also need $a_1(X) \geq \frac{X^{**}-X_1}{X^*}$ and $a_2(X) \geq \frac{X^{**}-X_2}{X^*}$ to guarantee that the buyers do not choose NB.

Moreover, at the boundary $X_2 = X^{**}$, $a_1(X) = \frac{X^{**}-X_1}{X^*}$ and $a_2(X) = 0$, $y_1(X) = \frac{1}{2}e^{\frac{r(1-A)}{2}(X_1-X_2)}$ and $y_2(X) = 1 - \frac{1}{2}e^{\frac{r(1-A)}{2}(X_1-X_2)}$ for $X_2 = X^{**}$ and $X_2 > X_1$.

Since $\frac{\partial V_2(X)}{\partial X_2} = \frac{r(1-A)}{2}y_2(X)$ at $X = (X_1, X^{**}+)$ and $r(y_2(X) - V_2(X)) = (1-2\frac{X_2-X_1}{X^*})\frac{\partial V_2(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2}$, then at $X = (X_1, X^{**})$,

$$\frac{r(1+A)}{2}y_2(X) = rV_2(X) + (1-2\frac{X^{**}-X_1}{X^*})\frac{\partial V_2(X)}{\partial X_1}.$$

Therefore, $V_2(X_1, X^{**})$ can be solved as a function of C.

If $f(X_1, X_2) \leq 0$, the buyer is indifferent among B_1, B_2 and NB, then $a_i(X) = \frac{-\alpha_0}{\alpha_1-\alpha_0} - \frac{\beta X_i}{\alpha_1-\alpha_0} = \frac{X^{**}-X_i}{X^*}$.

$$\begin{aligned}
V_1(X) &= e^{r\frac{1-A}{1+A}X_1}\phi(e^{\frac{1-A}{1+A}\frac{2X_1}{X^*}}(1-2\frac{X^{**}-X_2}{X^*})). \\
V_2(X) &= e^{r\frac{1-A}{1+A}X_2}\phi(e^{\frac{1-A}{1+A}\frac{2X_2}{X^*}}(1-2\frac{X^{**}-X_1}{X^*})).
\end{aligned}$$

If $f(X_1, X_2) = 0$, $\frac{\partial V_1(X)}{\partial X_1} + \frac{\partial V_2(X)}{\partial X_2} = \frac{r(1-A)}{2}$.

$\square$

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