A Model of Multiproduct Firm Growth: Online Appendix☆

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Abstract

This article studies optimal growth strategies of a multiproduct firm that invests in the qualities of different products, which have persistent effects on future payoffs and are modeled as a state variable of a stochastic game. We derive a unique Markov perfect equilibrium. At the early stage, the firm focuses on the product with higher quality, and may switch its specialization. If the quality of the specialized good is high enough, the firm diversifies to capture demands for all products. However, the firm may lose its focus on either product and get no demand, due to a moral hazard problem.

Keywords: firm dynamics; stochastic games; Markov perfect equilibrium; specialization; diversification; moral hazard

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Contents

1. Asymmetric Investment Cost

In this section, we study the asymmetric case where investment costs of two goods are different. Define $A_i \equiv \frac{\pi_i(1,1)-\pi_i(1,0)}{\pi_i(0,1)-\pi_i(0,0)}$, which captures the investment cost for good i: higher A_i implies lower investment cost for good i. Assumption 1.3 says that it is cheaper to invest in the quality of good 1, relative to good 2.

Assumption 1.1: $\pi_i(a_i, y_i)$ is decreasing in a_i and increasing in y_i; $\pi_i(1,1) > \pi_i(0,0)$.

Assumption 1.2: $\pi_i(a_i, y_i)$ is strict *submodular*: $\frac{\partial^2 \pi_i(a_i,y_i)}{\partial a_i \partial y_i} < 0$.

Assumption 1.3 (Asymmetry): $A_1 > A_2$ and $\pi_1(1,1) = \pi_2(1,1)$.

Assumption 1.4: $\frac{\partial u_i(a_i,y_i,X_i)}{\partial y_i}$ is increasing in a_i and X_i. The unique solution $a^*(X_i)$ to $\frac{\partial u_i(a_i,y_i,X_i)}{\partial y_i} = 0$ is linear and decreasing in X_i. Moreover, $a^*(0) \in (0,1)$.

Define $B \equiv \frac{A_2}{A_1}\frac{1-A_1}{1-A_2} \in (0,1)$ to capture the degree of asymmetric investment cost between two goods. $B = 1$ corresponds to the symmetric case. Define $\tilde{X}$ as a threshold of quality, above which it is a dominant strategy for buyers to buy. To describe the equilibria in a concise way, we partition the state space into three regions (see Figure A.1): a rewarding stage $\mathcal{R} \equiv \{X_1 + X_2 \geq X^*\}$ and two quality-building stages $\mathcal{B}_i = \{W_i(X_i) > W_{-i}(X_{-i})\} \cap \{X_1 + X_2 < X^*\}$, where $i = 1, 2$.[1] We consider the following *quality-cycle equilibria.*

1. If $X \in \mathcal{R}$, there is full demand for good 1: $y_1(X) = 1$, and a quality cycle for good 2 exists.
   (a) If $X_2 > \tilde{X}$, the firm chooses $a_2(X) = a^*(X_2)$ and the demand for good 2 is increasing in X_2: $y_2(X) = e^{-r\frac{1-A_2}{1+A_2}(\tilde{X}-X_2)}(1-B) + B$.
   (b) If $X_2 < \tilde{X}$, the firm does not invest in good 2: $a_2(X) = 0$ and there is no demand for good 2: $y_2(X) = 0$.
2. If $X \in \mathcal{B}_i$, then there is no demand for good $-i$: $y_{-i}(X) = 0$. There are X_1^* and X_2^* such that
   (a) If $X_i > X_i^*$, the firm invests only in good i: $a_i(X) = 1$ and there is full demand for good i: $y_i(X) = 1$.
   (b) If $X_i < X_i^*$, then the firm chooses $a_i(X) = a^*(X_i)$ and the demand for good i is increasing in X_i: $y_i(X) = e^{-r\frac{1-A_i}{1+A_i}(X_i^*-X_i)}$.

Theorem 1.1. *Under Assumptions* 1.1-1.4 *and* 1-1 *transition rule, there exist Markov perfect equilibria in the limit as* $\Delta \to 0$, *characterized by the quality-cycle equilibria.*

Theorem 1.1 shows that if the sum of two qualities is high enough: $X_1 + X_2 > X^*$, there is a rewarding stage $\mathcal{R}$ in which full demand for good 1 is guaranteed. Moreover, the firm focuses on investing in X_2 and there is a quality cycle for good 2. If $X_2 \leq \tilde{X}$, the firm chooses $a_2(X) = a^*(X_2)$ to make buyer 2 indifferent between buying and not buying, and the demand for good 2 is increasing in X_2. Then, the quality X_2 moves up and down stochastically. If $X_2 > \tilde{X}$, the firm starts to exploit high quality of good 2 by not investing in good 2, and thus X_2 depreciates deterministically. In the long run, X_2 is distributed in $[0, \tilde{X}]$.

[1]See the definition of $W_i(X_i)$, which is increasing in X_i, in Corollary A.1 of Appendix A .

If the sum of two qualities is low enough, $X_1 + X_2 \leq X^*$, there is a quality-building stage $\mathcal{B}_i$, in which the firm focuses on investing in good i with relative higher quality adjusted by asymmetric investment cost. As illustrated by Figure A.1, the region $\mathcal{B}_1$ is larger than the region $\mathcal{B}_2$, since good 1 is cheaper to invest in than good 2.[2]

2. Economies of Scope

In the previous sections, firm's stage-game total profit is the sum of profits from both goods. This section discusses the implication of economies of scope (Assumption 2.1). $\pi(a_1, y_1, y_2)$.

Assumption 2.1 (Economies of Scope): $\pi(a_1, y_1, y_2) > \pi(a_1, y_1, 0) + \pi(a_1, 0, y_2)$.

Assumption 2.2: $\pi(a_1, y_1, 0)$ is increasing in y_1 and decreasing in a_1.

Assumption 2.3: $\pi(a_1, y_1, 0)$ is submodular in (a_1, y_1): $\frac{\partial^2 \pi(a_1, y_1, y_2)}{\partial a_1 \partial y_1} < 0$.

Assumption 2.4: $\pi(a_1, y_1, y_2) = \pi(1 - a_1, y_2, y_1)$.

Assumptions 2.1-2.4 describe the firm's stage-game payoff. Assumption 2.1 says that the total profit from producing two goods is higher than the sum of profits from producing either good independently. Assumption 2.1 is actually a condition of the firm's cost structure, as illustrated by the following example. Define $\pi(a_1, y_1, y_2) = y_1 + y_2 - C(a_1, y_1, y_2)$, where $C(a_1, y_1, y_2)$ is the firm's total cost of producing y_1 and y_2. Assume that $C(a_1, y_1, y_2) = (\gamma + (1 - A)y_1)a_1 + (\gamma + (1 - A)y_2)a_2 - (\theta - 1)y_1 y_2$, where $\theta > 1$. Hence, Assumption 2.1 holds: $C(a_1, y_1, y_2) < C(a_1, y_1, 0) + C(a_1, y_1, 0)$. Notice that $\theta > 1$ also captures resource complementarity: a higher y_{-i} reduces the marginal cost of investing in good i.

Define $\theta \equiv \frac{\pi(1,1,1) - \pi(1,1,0)}{\pi(1,0,1) - \pi(1,0,0)} > 1$, which measures economies of scope between two goods: a higher θ means a higher degree of economies of scope. Define $A \equiv \frac{\pi(1,1,0) - \pi(1,0,0)}{\pi(0,1,0) - \pi(0,0,0)}$. By Assumption 2.3, $A \in (0, 1)$ captures *submodularity* of the firm's payoff: a higher A implies

[2] It is possible that $\mathcal{B}_2$ is empty.

a lower degree of submodularity. Define X^* such that $a^*(X^*) + a^*(0) = 1$ and $X^*_\theta = X^* - \frac{1}{r}\log\frac{2\theta}{1-A}$ as the quality threshold, above which the firm implements a complete specialization strategy.

Theorem 2.1. *Under Assumptions* 2.1-2.4, 1.4 *and* 1-1 *transition rule, there are Markov perfect equilibria in the limit as* $\Delta \to 0$*, characterized as follows:*

1. *If* $X_1 + X_2 > X^*$*, then the firm invests in both goods with high intensity:* $a_i(X) \geq a^*(X_i)$ *and there are full demands for both goods:* $y_1 = y_2 = 1$*.*
2. *If* $X_1 + X_2 < X^*$ *and* $X_i > X_{-i}$*, then there is no demand for good* $-i$*:* $y_{-i} = 0$*.*
   (a) *If* $X_i < X^*_\theta$*, then the firm chooses* $a_i(X) = a^*(X_i)$ *and the demand for good* i *is increasing in* X_i*:* $y_i(X) = e^{-r\frac{1-A}{1+A}(X^*_\theta - X_i)}$.
   (b) *If* $X_i \geq X^*_\theta$*, then the firm invests only in good* i*:* $a_i(X) = 1$*, and there is full demand for good* i*:* $y_i(X) = 1$*.*

Theorem 2.1 shows that economies of scope give the firm extra incentives to specialize, in the sense that there is a larger region of complete specialization: X^*_θ is decreasing in θ. The reason is as follows: if two goods display economies of scope (higher θ), then there is a larger payoff in the rewarding stage, and thus firm is more willing to invest only in the specialized good in order to reach the rewarding stage sooner. As a result, there is a larger region of complete specialization (lower X^*_θ). Furthermore, a higher degree of economies of scope also implies higher demand: $y_i(X)$ is increasing in θ, for $0 < X_i < X^*_\theta$.

3. Supermodularity

In the main model, the firm's investment cost of good i is increasing in demand y_i, which is captured by Assumption 1.2: the firm's payoff function is strictly submodular. In this section, we study the case of supermodularity as follows:

Assumption 3.1 (Supermodularity): $\pi_i(a_i, y_i)$ is *supermodular* in (a_i, y_i): $\frac{\partial^2 \pi_i(a_i,y_i)}{\partial a_i \partial y_i} \geq 0$.

Assumption 3.2: $\pi_1(a, y) = \pi_2(a, y)$, for $a, y \in [0, 1]$.

Theorem 3.1. *Under Assumptions* 1.1, 1.4, 3.1 *and* 3.2, *there exist monotone Markov perfect equilibria characterized as follows:*

1. *If $X_1 + X_2 > X^*$, the firm invests in both goods with high intensity: $a_i(X) \geq a^*(X_i)$, and there are full demands for both goods: $y_1(X) = y_2(X) = 1$.*
2. *If $X_1 + X_2 \leq X^*$ and $X_i > X_{-i}$, then the firm invests only in good i: $a_i(X) = 1$, there is full demand for good i: $y_i(X) = 1$ and no demand for good $-i$: $y_{-i}(X) = 0$.*

In the rewarding stage ($X_1 + X_2 > X^*$), the firm invests in a "balanced way" to capture full demands for both goods, which is exactly the same as the case of submodularity. Under supermodularity, if $X_1 + X_2 \leq X^*$ and $X_i > X_{-i}$, there is a specialization stage where the firm completely specializes in good i, for the following reason. By supermodularity, $y_i(X) \geq y_{-i}(X)$ (*monotonicity*) implies that the marginal cost of investing in the specialized good is less than the other good. Together with the long-run benefit from investing in specialized good (that is, to enter the rewarding stage earlier), the firm strictly prefers to invest in the specialized good, which leads to full demand for the specialized good. In total, the firm's motivation to specialize is so strong that there is neither an exploration stage nor an aggressive growth stage.

4. Non-linearity

In this section, we relax the linearity of $a^*(X_i)$ by allowing non-linearity, as is stated by Assumption 4.1. Notice that higher $a^*(X_i)$ implies lower benefit for buyer i from buying good i. Therefore, the concavity (convexity) of $a^*(X_i)$ captures the idea that buyer i's marginal benefit from buying good i is increasing (decreasing) in X_i.

Assumption 4.1 (Non-linearity): The unique solution $a^*(X_i)$ to $\frac{\partial u_i(a_i, y_i, X_i)}{\partial y_i} = 0$ is concave (or convex) and decreasing in X_i. Moreover, $a^*(0) \in (0, 1)$.

Define the rewarding stage $\mathcal{R} \equiv \{X_1 + X_2 \geq X_N\}$, where X_N is the smallest number to satisfy that $\{a^*(X_1) + a^*(X_2) \leq 1\} \subset \{X_1 + X_2 \geq X_N\}$.[3] Define $X_N^* = X_N - \frac{1}{r} \log \frac{2}{1-A}$ as a

[3]In Figure A.2, the dashed line is the curve that $\{(X_1, X_2) : a^*(X_1) + a^*(X_2) = 1\}$, and the region above

quality threshold, above which the firm implements a complete specialization strategy.

Theorem 4.1. *Under Assumptions* 2.1-2.3, 4.1 *and* 1-1 *transition rule, there exists a monotone Markov perfect equilibrium if r is small enough:*

1. *If $X \in \mathcal{R}$, the firm invests in both goods with high intensity: $a_i(X) \geq a^*(X_i)$ and there are full demands for both goods: $y_1(X) = y_2(X) = 1$.*
2. *If $X \notin \mathcal{R}$ and $X_i > X_{-i}$, there is no demand for good $-i$: $y_{-i}(X) = 0$.*
   (a) *If $X_i < X_N^*$, the firm chooses $a_i(X) = a^*(X_i)$ and demand $y_i(X)$ is increasing in X_i: $y_i(X) = e^{-r\frac{1-A}{1+A}(X_N^* - X_i)}$.*
   (b) *If $X_i \geq X_N^*$, the firm invests only in good i: $a_i(X) = 1$ and there is full demand for good i: $y_i(X) = 1$.*

We make the following three observations. First, the boundary of the rewarding stage is linear, which is independent of the curvature $a^*(X_i)$, since it is the direction of the transition rule that determines the shape of the rewarding stage.

Second, the concavity of $a^*(X_i)$ does not influence the size of the rewarding stage. If $a^*(X_i)$ is concave in X_i, then $\{(X_1, X_2) : a^*(X_1) + a^*(X_2) \leq 1\}$ is a convex set. By the definition of X_N, we can see $X_N = X^*$ in Figure A.2. Therefore, the size of the rewarding stage is independent of the concavity of $a^*(X_i)$.

Finally, the convexity of $a^*(X_i)$ impacts the size of the rewarding stage. If $a^*(X_i)$ is convex in X_i, then the region $\{(X_1, X_2) : a^*(X_1) + a^*(X_2) \geq 1\}$ is a convex set. Define $\hat{X}^*$ such that $a^*(\frac{\hat{X}^*}{2}) = \frac{1}{2}$, and by the definition of X_N, we can see $X_N = \hat{X}^* > X^*$ in Figure A.2. Therefore, if buyer i's marginal benefit from buying good i is decreasing in X_i ($a^*(X_i)$ is convex), the size of the rewarding stage shrinks, compared with the case of constant marginal benefit ($a^*(X_i)$ is linear).

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the dashed curve is $\{(X_1, X_2) : a^*(X_1) + a^*(X_2) \leq 1\}$.

5. Spillover Effect

In previous sections, we assume that the marginal benefit of buyer i only depends on a_i and X_i, independent of the other quality X_{-i}. As a result, the cutoff strategy $a^*(X_i)$ only depends on X_i, as stated by Assumption 1.4. In this section, we relax this assumption by allowing "spillover effect" to take place.

Assumption 5.1 (Spillover effect): The unique solution $a_i^*(X_1, X_2)$ to $\frac{\partial u_i(a_i, y_i, X_1, X_2)}{\partial y_i} = 0$ is linear in X_i and X_{-i}, and decreasing in X_i; moreover, $|\frac{\partial a_i^*(X_1,X_2)}{\partial X_i}| > |\frac{\partial a_i^*(X_1,X_2)}{\partial X_{-i}}|$.

Assumption 5.1 defines spillover effect in terms of $a_i^*(X_1, X_2)$, which is a function of not only X_i but also X_{-i}. If $a^*(X_1, X_2)$ is decreasing (increasing) in X_{-i}, the quality of good $-i$ has a positive (negative) impact on the buyer i's willingness to pay, which is defined as positive (negative) spillover effect. Furthermore, we only focus on the economically interesting case that X_i has a larger impact on buyer i's willingness to pay than X_{-i}.

Define the rewarding stage $\mathcal{R} \equiv \{X_1 + X_2 \geq X_S\}$, which X_S is such that $X_1 + X_2 \geq X_S$ is equivalent to $a_1^*(X_1, X_2) + a_2^*(X_1, X_2) \leq 1$. Define $X_S^* = X_S - \frac{1}{r} \log \frac{2}{1-A}$ as a quality threshold, above which the firm implements a complete specialization strategy.

Theorem 5.1. *Under Assumptions* 2.1-2.3, 5.1 *and* 1-1 *transition rule, there exists a monotone Markov perfect equilibrium:*

1. *If $X \in \mathcal{R}$, the firm invests in both goods with high intensity: $a_i(X) \geq a_i^*(X_1, X_2)$ and there are full demands for both goods: $y_1(X) = y_2(X) = 1$.*
2. *If $X \notin \mathcal{R}$ and $X_i > X_{-i}$, there is no demand for good $-i$: $y_{-i}(X) = 0$.*
   (a) *If $X_i < X_S^*$, the firm chooses $a_i(X) = a_i^*(X_1, X_2)$ and demand $y_i(X)$ is increasing in X_i: $y_i(X) = e^{-r\frac{1-A}{1+A}(X_S^* - X_i)}$.*
   (b) *If $X_i \geq X_S^*$, the firm invests only in good i: $a_i(X) = 1$ and there is full demand for good i: $y_i(X) = 1$.*

Theorem 5.1 says that the equilibrium is similar to the case without the spillover effect, except that the sizes of the rewarding stage are different. We find that there is a larger size of rewarding stage, if and only if there is a positive spillover effect. Consequently, the firm has stronger incentives to specialize outside the rewarding stage.

6. Price-quality effect

In previous sections, price is independent of the product's quality. In this section, we relax this assumption, and assume that the quality of good $i = 1, 2$ directly improves the profitability of the firm, i.e., price-quality effect. Define $\pi_i^{\lambda_i}(a_i, y_i, X_i)$ and $u_i^{\lambda_i}(a_i, y_i, X_i)$ as the firm's profit from good i, and buyer i's utility function respectively. The following assumption (Assumption 6.1) captures the price-quality effect, which is measured by parameter $\lambda_i \geq 0$. Note that $\lambda_i = 0$ corresponds to the main model without price-quality effect.

Assumption 6.1 (price-quality effect): $\pi_i^{\lambda_i}(a_i, y_i, X_i) = \pi^0(a_i, y_i, X_i) + \lambda_i X_i y_i$ and $u_i^{\lambda_i}(a_i, y_i, X_i) = u_i^0(a_i, y_i, X_i) - \lambda_i X_i y_i$, where $\lambda \geq 0$.

A general form of the firm's profit from good i is as follows: $\pi_i(1, 1) = A - \gamma + \lambda_i X_i$, $\pi_i(0, 1) = 1 + \lambda_i X_i$, $\pi_i(1, 0) = -\gamma$ and $\pi_i(0, 0) = 0$, for $i = 1, 2$. Therefore, the firm's profit from good i under investment a_i and demand y_i is $(1 + \lambda_i X_i)y_i - (\gamma + (1 - A)y_i)a_i$, where the price of good i is $1 + \lambda_i X_i$, which is increasing in the quality X_i. A general form of buyer i's utility function is $u_i^{\lambda_i}(a_i, y_i, X_i) = (u_0 + (u_1 - u_0)a_i + (\alpha - \lambda_i)X_i)y_i$, where $u_0 < 0 < u_1$. Notice that αX_i represents the total gain from quality X_i. Define $a_{\lambda_i}^*(X_i)$ as the firm's investment intensity for good i so that the buyer i is indifferent between buying or not. It is trivial that $a_{\lambda_i}^*(X_i) = \frac{-u_0}{u_1 - u_0} - \frac{\alpha - \lambda_i}{u_1 - u_0} X_i$.

6.1. *Exogenous price-quality effect*

We first assume that $\lambda_1 = \lambda_2 \equiv \lambda > 0$, which measures the firm's market power, is exogenously given. Define the rewarding stage $\mathcal{R}_\lambda \equiv \{X_1 + X_2 \geq X_\lambda\}$, which X_λ is such that $X_1 + X_2 \geq X_\lambda$ is equivalent to $a_\lambda^*(X_1) + a_\lambda^*(X_2) \leq 1$. Define $X_\lambda^* = X_\lambda - \frac{1}{r} \log \frac{1 - \frac{\lambda}{r}}{\frac{1-A}{2} - \frac{\lambda}{r}}$ as a quality threshold, above which the firm implements a complete specialization strategy.[4]

[4] $X_\lambda = \frac{-u_0 - u_1}{\alpha - \lambda}$. The definition of X_λ^* makes sense only if $\lambda < \frac{r(1-A)}{2}$.

We give a complete characterization of the equilibrium as follows:

Theorem 6.1. *Under Assumptions* 1.1*,* 1, 2*,* 1.4*,* 2.4*,* 6.1*,* 1-1 *transition rule and* $\lambda \leq r$*, there exists a monotone Markov perfect equilibrium:*

1. If $X \in \mathcal{R}_\lambda$*,* $a_1(X) \geq a^*_\lambda(X_1)$*,* $a_2(X) \geq a^*_\lambda(X_2)$ *and* $y_1(X) = y_2(X) = 1$*.*

2. If $X \notin \mathcal{R}_\lambda$ *and* $X_i > X_{-i}$*, there is no demand for good* $-i$*:* $y_{-i}(X) = 0$*.*

  (a) If $\lambda < \frac{r(1-A)}{2}$*, then*

    i. If $X_i < X^*_\lambda$*,* $a_i(X) = a^*_\lambda(X_i)$ *and* $y_i(X) = (\frac{1+A+2\lambda X_1}{1+A+2\lambda X^*_\lambda})^{\frac{r(1-A)-2\lambda}{2\lambda}}$*.*

    ii. If $X_i \geq X^*_\lambda$*,* $a_i(X) = 1$ *and* $y_i(X) = 1$*.*

  (b) If $\lambda \geq \frac{r(1-A)}{2}$*, then* $a_i(X) = 1$ *and* $y_i(X) = 1$*.*

Theorem 6.1 shows that the properties of the equilibrium with price-quality effect remain similar to those without the price-quality effect. However, as λ increases, there are three quantitative impacts on the firm's life-time payoff: (1) the stage-game payoff of the firm directly increases by $\lambda X_i y_i$ in each period; (2) the firm spends more time on the specialization stage: $X_\lambda - X^*_\lambda$ is increasing in λ; (3) there is a delay before the firm enters the rewarding stage (X_λ is increasing in λ), since the consumers are less willing to buy due to the fact that the terms of trade have been worsened. The first and second effects improve the firm's payoff, while the third effect harms the firm's payoff.

6.2. Endogenous price-quality effect

In the first scenario, we assume that there is an initial period in which the firm is given the opportunity of choosing a constant $\lambda_1 = \lambda_2 = \lambda \in [0, \alpha]$ for all the future periods, where $\alpha > 0$ is the total gain from quality upgrade for both goods. According to the analysis above, if the third effect dominates the first and the second effects, the firm's optimal choice is $\lambda = 0$ at a certain state X.[5]

In the second scenario, we extend the analysis by assuming that λ_i is endogenously determined by the firm in each period. Formally, in each period, the firm first chooses

[5]See the proof of Theorem 6.1 (Step 3) for details.

$\lambda_i \in [0, \alpha]$. After λ_i is chosen and observed by the buyers, the firm and two buyers play a product-choice game simultaneously, as illustrated in the main model. The next result (Theorem 6.2) demonstrates a Markov perfect equilibrium under endogenous choice of λ_i. Define that $X^{**} = X^* - \frac{1}{r} \log \frac{1-\frac{\alpha}{r}-\alpha X^*}{\frac{1-A}{2}-\frac{\alpha}{r}}$.[6]

Theorem 6.2. *Under Assumptions* 1.1, 1, 2, 1.4, 2.4, 6.1, *and* 1-1 *transition rule, there exists a monotone Markov perfect equilibrium:*

1. *If* $X^* \leq \frac{1}{\alpha} - \frac{1}{r}$, *then*
   (a) *If* $X_1 + X_2 > X^*$, $a_1(X) = a^*_{\lambda_1}(X_1)$ *and* $a_2(X) = a^*_{\lambda_2}(X_2)$, *where* $\lambda_1 X_1 + \lambda_2 X_2 = \alpha(X_1 + X_2 - X^*)$, *and* $y_1(X) = y_2(X) = 1$.
   (b) *If* $X_1 + X_2 \leq X^*$ *and* $X_i > X_{-i}$, *there is no demand for good* $-i$: $y_{-i}(X) = 0$.
      i. *If* $\alpha < \frac{r(1-A)}{2}$, *then*
         A. *If* $X_i < X^{**}$, *then* $\lambda_i = 0$, $a_i(X) = a^*_0(X_i)$ *and* $y_i(X) = e^{-r\frac{1-A}{1+A}(X^{**}-X_1)}$.
         B. *If* $X_i \geq X^{**}$, *then* $\lambda_i = \alpha$, $a_i(X) = 1$ *and* $y_i(X) = 1$.
      ii. *If* $\alpha \geq \frac{r(1-A)}{2}$, *then* $\lambda_i = \alpha$, $a_i(X) = 1$ *and* $y_i(X) = 1$.
2. *If* $X^* > \frac{1}{\alpha} - \frac{1}{r}$, *then at states* $X_i > X_{-i}$, $\lambda_i = \alpha$, $a_i(X) = 1$ *and* $y_i(X) = 1$.

Theorem 6.2 characterizes the equilibrium if the firm has the opportunity to choose the price in each period. If X^* is low enough (it is easy to convince both buyers to buy), there is a rewarding stage, a specialization stage and an exploration stage, similar to the equilibrium without price-quality effect in the main model. In addition, we analyze the price dynamics: (1) in the exploration stage, the firm sets the price of the specialized good i at the lowest level: $\lambda_i = 0$, in order to capture a higher demand in each period; (2) in the specialization stage, the firm sets the price of the specialized good i at the highest level: $\lambda_i = \alpha$, since there is full demand for good i, independent of the price level; (3) in the rewarding stage, the firm enjoys the benefits from higher quality by setting higher prices: the total price $p_1 + p_2$, which is equal to $2 + \alpha(X_1 + X_2 - X^*)$, is increasing in the total quality $X_1 + X_2$.

If X^* is high enough (it is difficult to convince both buyers to buy), it is not the firm's

[6]Notice that the above definition makes sense if $\frac{\alpha}{r} < \min\{\frac{1-A}{2}, 1 - \alpha X^*\}$.

optimal choice to diversify and enjoy the reward from both goods, since the firm needs to lower the price of the specialized good i, in order to capture both demands. To be specific, at state $(X^*, 0)$, the firm sets $\lambda_1 = 0$ in order to capture both demands, but the firm chooses $\lambda_1 = \alpha$ by targeting only good 1. Therefore, the cost of diversification comes form a price drop: αX^*, and consequently the firm may not pursue diversification if the benefit from adding a new product is dominated by the cost of diversification, which holds if X^* is high enough.

7. Appendix

Proof of Theorem 1.1:

For simplicity, we assume that $\pi_i(0,0) = \pi_i(1,0)$, which means that the firm gets zero payoff from good i if buyer i does not buy.

Proof. **Case 1**: $X_1 + X_2 \geq X^*$.

We construct an equilibrium where the firm chooses $a_2(X) = a^*(X_2)$ so that buyer 2 is indifferent between buying and not buying. By $X_1 + X_2 \geq X^*$ and $a_2(X) = a^*(X_2)$, we know that $a_1(X) > a^*(X_1)$, thus buyer 1 buys with probability one. Moreover, the equilibrium only depends on X_2, thus we denote $v(X_2) \equiv V(X_1, X_2)$.

(1) If $X_2 > \tilde{X}$, then $a_1(X) = 0,\ a_2(X) = 0$ and $y_1(X) = y_2(X) = 1$.

(2) If $\Delta \leq X_2 \leq \tilde{X}$, the firm is indifferent between I_2 and I_1.

We have $v(X_2) = \pi_1(1,1) + \pi_2(0,1)y_2(X_2) + \delta v(X_2 - \Delta) = \pi_1(0,1) + \pi_2(1,1)y_2(X_2) + \delta v(X_2 + \Delta)$. Define $Y_2(X_2) \equiv (\pi_2(0,1) - \pi_2(1,1))y_2(X_2) - (\pi_1(0,1) - \pi_1(1,1))$ and denote $Y_k = Y_2(k\Delta)$. Define $\tilde{X} = \tilde{K}\Delta$. For $2 \leq k \leq \tilde{K}$, $Y_k = \frac{1}{\delta}(1 - A_2)Y_{k-1} + A_2 Y_{k-2}$. By similar argument, we can show that $Y_1 = (\frac{1}{\delta}(1 - A_2) + 1)Y_0$. Moreover, for $k > \tilde{K}$, $Y_k = \pi_2(0,1) + \pi_1(1,1) - \pi_2(1,1) - \pi_1(0,1) > 0$. In all, by the second-order difference equation and two boundary conditions, we can solve the sequence $\{Y_k\}_{t=2}^{+\infty}$.

In the limit $\Delta \to 0$, if $X_2 \in (0, \tilde{X}]$, then $y_2(X_2) = e^{-r\frac{1-A_2}{1+A_2}(\tilde{X} - X_2)}(1 - B) + B$, and $v(X_2) = \pi_1(1,1) + \pi_2(0,1)(\frac{1+A_2}{2}e^{-r\frac{1-A_2}{1+A_2}(\tilde{X} - X_2)}(1 - B) + B)$, where $B = \frac{\pi_1(0,1) - \pi_1(1,1)}{\pi_2(0,1) - \pi_2(1,1)} \in (0,1)$.

Case 2: $X_1 + X_2 < X^*$.

Define X_1^* and X_2^* as $X_1^* = \max\{X^* - \frac{1}{r}\log\frac{(1+A_2)e^{-r\frac{1-A_2}{1+A_2}\tilde{X}}(1-B)+2B}{1-A_2}, 0\}$, $X_2^* = \max\{X^* - \frac{1}{r}\log\frac{(1+A_2)e^{-r\frac{1-A_2}{1+A_2}(\tilde{X}-X^*)}(1-B)+2B}{1-A_1}, 0\}$. Define

$$W_i(X_i) = \begin{cases} \pi_i(0,1)\frac{1+A_i}{2}e^{-r\frac{1-A_i}{1+A_i}(X_i^*-X_i)} & X_i \in [0, X_i^*] \\ \pi_i(0,1)(A_i + e^{-r(X_i^*-X_i)}\frac{1-A_i}{2}) & X_i \in (X_i^*, X^*] \end{cases}$$

By the above definition, it is trivial that $W_1(X^*) = v(0)$ and $W_2(X^*) = v(X^*)$.

Step 1: Show that $W_2(0) < W_1(X^*)$ and $W_1(0) < W_2(X^*)$.

By Assumption 4.1, $W_2(0) < W_1(X^*)$ is equivalent to $(1-e^{-r\frac{1-A_2}{1+A_2}\tilde{X}})(1-B) < 1$, which is true. Furthermore, $W_1(0) \le W_1(X_1^*) = \frac{\pi_1(1,1)+\pi_1(0,1)}{2} < \frac{\pi_2(1,1)+\pi_2(1,1)}{2} = W_2(X_2^{**}) < W_2(X^*)$, as $W_i(X_i)$ is strictly increasing in $(0, X^*]$.

Step 2: Show that $V(X^*, 0) = W_1(X^*)$ and $V(0, X^*) = W_2(X^*)$.

Because $W_2(0) < W_1(X^*)$, then at state $X = (X^*, 0)$, $a_1(X) = y_1(X) = 1$ and $a_2(X) = y_2(X) = 0$. By Case 1, for all $X_1 > X^*$, $V(X_1, 0) = v(0)$. By *value matching*, $V(X^*, 0) = v(0) = W_1(X^*)$.

As $W_1(0) < W_2(X^*)$, at state $X = (0, X^*)$, $a_2(X) = y_2(X) = 1$, $a_1(X) = y_1(X) = 0$. By Case 1, for all $X_2 > X^*$, $V(0, X_2) = v(X_2)$. By *value matching*, $V(0, X^*) = v(X^*) = W_2(X^*)$.

Step 3: If $X_1 + X_2 \le X^*$ and the firm only focuses on investing in good i: $a_i(X) \ge a^*(X_i)$, $y_i > 0$ and $y_{-i} = 0$, then $W_i(X_i)$ is the firm's value function and the players' behavior is described as follows:

(1) If $0 < X_i \le X_i^*$, then $a_i(X_i) = a^*(X_i)$, $y_i(X_i) = e^{-r\frac{1-A_i}{1+A_i}(X_i^*-X_i)}$, $y_j(X) = 0$.

(2) If $X_i^* < X_i \le X^*$, then $a_i(X_i) = 1$, $y_i(X_i) = 1$, $y_j(X) = 0$.

For $X_i^* < X_i \le X^*$, $W_i(X_i) = (1-e^{-r(X^*-X_i)})\pi_i(1,1) + e^{-r(X^*-X_i)}W_i(X^*) = \pi_i(0,1)(A_i + e^{-r(X_i^*-X_i)}\frac{1-A_i}{2})$.

For $0 < X_i \le X_i^*$, $W_i(X_i) = e^{-r\frac{1-A_i}{1+A_i}(X_i^*-X_i)}W_i(X_i^*) = \pi_i(0,1)\frac{1+A_i}{2}e^{-r\frac{1-A_i}{1+A_i}(X_i^*-X_i)}$.

Step 4: Show that the firm's equilibrium value function $V(X_1, X_2) = \max\{W_1(X_1), W_2(X_2)\}$.

Assume that $W_i(X_i) > W_{-i}(X_{-i})$. By choosing $a_i(X) \geq a^*(X_i)$ and $y_{-i}(X) = 0$, the firm gets continuation value $W_i(X_i)$. By choosing $a_{-i}(X) \geq a^*(X_{-i})$ and thus $y_i(X) = 0$, the firm gets continuation payoff $W_{-i}(X_{-i})$. Therefore, the firm chooses $a_i(X) \geq a^*(X_i)$ and thus $y_{-i}(X) = 0$. Indeed, the firm invests in X_i and the equilibrium will be characterized by Step 3 where $y_{-i}(X) = 0$.

Consider the state where $W_1(X_1) = W_2(X_2)$. Because the continuation payoff of investing in either good is the same, in order for the firm to be indifferent between I_1 and I_2, both buyers choose $y_i(X_1, X_2) = 0$. Then, the firm chooses $a_i(X) \leq a^*(X_i)$. □

Corollary A.1: Under the same assumptions as in Theorem 1.1, $B_2 \equiv \{W_2(X_2) \geq W_1(X_1)\}$ is equivalent to $X_2 \geq f(X_1)$:

$$f(X_1) = \begin{cases} \frac{1-A_1}{1-A_2}\frac{1+A_2}{1+A_1}X_1 + \underline{X}_2 & X_1 \in [0, X_1^{**}) \\ \frac{1}{r}\frac{1+A_2}{1-A_2}\ln(1 + \frac{1-A_1}{1+A_1}(e^{r(X_1 - X_1^{**})} - 1)) + \bar{X}_2 & X_1 \in [X_1^{**}, \bar{X}_1) \\ X_1 + X_2^{**} - \bar{X}_1 & X_1 \geq \bar{X}_1. \end{cases}$$

where $\underline{X}_2 = X_2^{**} - \frac{1}{r}\frac{1+A_2}{1-A_2}\ln(\frac{1+A_2}{1+A_1}\frac{A_1}{A_2}) - \frac{1-A_1}{1-A_2}\frac{1+A_2}{1-A_2}X_1^{**}$, $\bar{X}_2 \equiv X_2^{**} - \frac{1}{r}\frac{1+A_2}{1-A_2}\ln(\frac{1+A_2}{1+A_1}\frac{A_1}{A_2})$, $\bar{X}_1 = X_1^{**} + \frac{1}{r}\ln(\frac{1}{B})$.

Proof. By Theorem 1.1, we have

$$W_i(X_i) = \begin{cases} \pi_i(0,1)\frac{1+A_i}{2}e^{-r\frac{1-A_i}{1+A_i}(X_i^* - X_i)} & X_i \in [0, X_i^*] \\ \pi_i(0,1)(A_i + e^{-r(X_i^* - X_i)}\frac{1-A_i}{2}) & X_i \in (X_i^*, X^*] \end{cases}$$

By $W_1(X_1) = W_2(X_2)$, the result follows.

□

Proof of Theorem 2.1:

Proof. **Step 1:** $X_1 + X_2 > X^*\Delta$.

If $y_1(X) = y_2(X) = 1$, the firm gets the same payoff in the current period from I_1 or I_2 as the firm's payoff is symmetric: $\pi(1, 1, 1) = \pi(0, 1, 1)$. Therefore, it is optimal for the firm to choose $a_i(X) \geq a^*(X_i)$ for $i = 1, 2$.

Step 2: Show that it is optimal for the firm to choose to invest in good 1 at state $X_1 \in (X_\theta^*, X^*]$ and $X_1 > X_2$, i.e. $\delta(V(X_1 + 2\Delta) - V(X_1) > \pi(0, 1, 0) - \pi(1, 1, 0)$, where the firm's value function $V(X)$ only depends on X_1.

If $X_1 > X_2$, $y_2(X) = 0$ and the payoff is $\pi(a_1, y_1, 0)$. Show that $a_1(X) = y_1(X) = 1$ for all $X_\theta^* < X_i \leq X^*$. Denote $X_1 = k\Delta$, then for $k\Delta > X_\theta^*$, $V(X_1) = (1 - \delta^{X^*-k})\pi(1, 1, 0) + \delta^{X^*-k}V(X^*) = (1 - \delta^{X^*-k})\pi(1, 1, 0) + \delta^{X^*-k}\pi(1, 1, 1)$. By $\delta^{X^*-k} > \frac{1-A}{(1+\delta)\theta}$ for all $k\Delta \geq X_\theta^*$, we can show that $\delta(V(X_1 + 2\Delta) - V(X_1) > \pi(0, 1, 0) - \pi(1, 1, 0)$. Similarly, we can show that $\delta(V(X_\theta^* + 2\Delta) - V(X_\theta^*)) > \pi(0, 1, 0) - \pi(1, 1, 0)$. In all, for all $X_\theta^* \leq X_1 \leq X^*$, it is optimal for the firm to choose to invest in good 1.

The rest proofs for the case of $\theta > 1$ are the same as the case of complementarity: $\theta = 1$.

$\square$

Proof of Theorem 3.1

Proof. At state $X_1 + X_2 \geq X^*$, both buyers buys: $y_i(X) = 1$ and the firm randomizes: $a_i(X) \geq a^*(X_i)$ for $i = 1, 2$, by Proposition 3.1. We only need to check the case of $X_1 + X_2 < X^*$ and $X_1 > X_2$. By Proposition 3.1, we can show that $y_2(X) = 0$, and thus we only need to show that $a_1(X) = y_1(X) = 1$. First, consider the game between two buyers and a short-run firm. If $y_1(X) = 1$, then by *supermodularity*, the best response of a short-run firm is to invest in good 1: the payoff of investing in good 1 is $\pi_1(1, 1) + \pi_2(0, 0)$, which is not less than the payoff of investing in good 2: $\pi_1(0, 1) + \pi_2(1, 0)$. If $a_1(X) = 1$, the best response of buyer 1 is to buy: $y_1(X) = 1$. Next, we add long-run incentive to the firm. With long-run incentives, the firm has even larger incentive to invest in good 1, because there is an extra benefit of entering the rewarding stage earlier if $X_1 > X_2$.

$\square$

Proof of Theorem 4.1

Proof. The proof is the same as the one in the case of linearity. The only difference is that r needs to be small, so that X_N^* is small enough that the region $\{a^*(X_1) + a^*(X_2) \leq 1\} \cap \{X_1 + X_2 \leq X_N\} \cap \{X_i > X_{-i}\}$ belongs to the region $X_i > X_N^*$. This is because in the region $\{a^*(X_1) + a^*(X_2) \leq 1\} \cap \{X_i > X_{-i}\}$, it is necessary that $y_i(X) = 1$ in any *monotone* Markov perfect equilibrium, which holds for $X_i > X_N^*$. □

Proof of Theorem 5.1

Proof. Notice that the shape of the rewarding stage is the same as that in the case of no spillover effect, so the proof is also the same, except that the sizes of the rewarding stage in two cases differ.

□

Proof of Theorem 6.1

Proof. **Step 1:** $X_1 + X_2 > X_\lambda$.

If $X_1 + X_2 > X_\lambda$, the demand for the good with higher quality is full, and the demand for the other good is positive. Assume that $X_1 > X_2$, then $y_1 = 1$ and the firm chooses $a_2 = a_\lambda^*(X_2)$ so that $y_2 \in [0, 1]$. The indifferent condition of the firm implies that $V(X_1, X_2) = (1-\delta)(A+\lambda X_1-\gamma+(1+\lambda X_2)y_2)+\delta V(X_1+\Delta, X_2-\Delta) = (1-\delta)(1+\lambda X_1+(A+\lambda X_2)y_2-\gamma)+\delta V(X_1-\Delta, X_2+\Delta)$. Therefore, $\delta(V(X_1+\Delta, X_2-\Delta)-V(X_1-\Delta, X_2+\Delta)) = (1-\delta)(1-A)(1-y_2)$. If $\Delta \to 0$, then $\frac{\partial V}{\partial X_1} - \frac{\partial V}{\partial X_2} = \frac{r(1-A)(1-y_2)}{2}$ and $rV(X_1, X_2) = r(1 + A - \gamma + \lambda(X_1 + X_2)) - \frac{1+A+2\lambda X_2}{1-A}(\frac{\partial V}{\partial X_1} - \frac{\partial V}{\partial X_2})$. A solution is $V(X_1, X_2) = 1 + A - \gamma + \lambda(X_1 + X_2),\ y_2(X_1, X_2) = 1$.

Step 2: $X_1 + X_2 \leq X_\lambda$ and $X_1 > X_2$.

In the region $X_1 + X_2 \leq X_\lambda$, the demand for the goods with lower quality is zero. Therefore, we need to focus on the quality dynamics of the good with higher quality. Assume that $X_1 > X_2$, then $y_2 = 0$ and the continuation value only depends on X_1, by investing with intensity a_i, the profit from good 1 is $(1 + \lambda X_1)y_1 - a_1(\gamma + (1 - A)y_1)$ and the profit from good 2 is $-\gamma(1-a_1)$. Then, $V(X_1) = \max_{a_1}(1-\delta)((1+\lambda X_1)y_1 - (\gamma+(1-A)y_1)a_1 - \gamma(1-a_1)) + \delta(a_1 V(X_1+\Delta) + (1-a_1)V(X_1-\Delta))$, where $1-\delta = r\Delta$.

The existence of an interior solution for $a_i \in (0,1)$ implies that $\delta(V(X_1+\Delta) - V(X_1-\Delta)) = (1-\delta)(1-A)y_1$. Therefore, combing with the fact that $V(X_1) = (1-\delta)((A+\lambda X_1)y_1 - \gamma) + \delta V(X_1+\Delta)$, in the limit that $\Delta \to 0$, we have $y_1(X_1) = \frac{2}{r(1-A)}V'(X_1)$, $rV(X_1) = \frac{1+A+2\lambda X_1}{1-A}V'(X_1) - r\gamma$. The value function is $V(X_1) = C(1+\frac{2\lambda}{1+A}X_1)^{\frac{r(1-A)}{2\lambda}} - \gamma$ and the demand function is $y_1(X_1) = \frac{2C}{1+A}(1+\frac{2\lambda}{1+A}X_1)^{\frac{r(1-A)-2\lambda}{2\lambda}}$.

If $\lambda > \frac{r(1-A)}{2}$, then $y_1(X_1)$ is decreasing in X_1, then it is obvious that it is a dominant strategy for the firm to invest in good 1: $a_1 = 1$ and consequently $y_1 = 1$, for any $X_1 > X_2$. Otherwise, assume that $y_1(X_1) < 1$ for some $X_1 < X_\lambda$, and then $y_1(X_\lambda) < y_1(X_1) < 1$, a contradiction to the matching condition that $y_1(X_\lambda -) = y_1(X_\lambda +) = 0$.

If $\lambda < \frac{r(1-A)}{2}$, then $y_1(X_1)$ is increasing in X_1. We show that there is a threshold $X^*_\lambda < X_\lambda$ such that

- If $X_1 \in (X^*_\lambda, X_\lambda)$, the firm strictly prefers to invest in good 1;
- If $X_1 \in (0, X^*_\lambda)$, the firm invests with $a_1 = a^*_\lambda(X_1)$.

If $X_1 \in (X^*_\lambda, X_\lambda)$, the limit condition is $rV(X_1) = r(A+\lambda X_1 - \gamma) + V'(X_1)$. Together with a boundary condition $V(X_\lambda, 0) = 1+A-\gamma+\lambda X_\lambda$ at $(X_\lambda, 0)$, the solution is that for $X_1 \in (X^*_\lambda, X_\lambda)$, $V(X_1) = 1+A-\gamma+\lambda X_1 - (1-\frac{\lambda}{r})(1-e^{-r(X_\lambda - X_1)})$. A necessary condition is $\lambda \leq r$.

If $X_1 \in (0, X^*_\lambda)$, the solution is $V(X_1) = C(1+\frac{2\lambda}{1+A}X_1)^{\frac{r(1-A)}{2\lambda}} - \gamma$ (which is shown above). To pin down the constant C and X^*_λ, we need to match $V(X^*_\lambda -) = V(X^*_\lambda +)$ and $y_1(X^*_\lambda) = 1$. The solution is $X^*_\lambda = X_\lambda - \frac{1}{r}\log\frac{1-\frac{\lambda}{r}}{\frac{1-A}{2}-\frac{\lambda}{r}}$. As result, for $0 < X_1 < X^*_\lambda$, $y_1(X_1) = (\frac{1+A+2\lambda X_1}{1+A+2\lambda X^*_\lambda})^{\frac{r(1-A)-2\lambda}{2\lambda}}$.

Step 3: The optimal choice of λ.

From Step 2, if the state X is in the specialization stage with $X_1 > X_2$, then the continuation payoff is $V(X_1) = 1+A-\gamma+\lambda X_1 - (1-\frac{\lambda}{r})(1-e^{-r(X_\lambda - X_1)})$. Now the firm is given the opportunity to choose λ for all future periods. Therefore, the firm choose λ to maximize $V(X_1)$. If $\frac{dX_\lambda}{d\lambda}$ is large enough, then $\frac{dV(X_1)}{d\lambda} = \frac{dV(X_1)}{d\lambda} = X_1 + \frac{1}{r}(1 - e^{-r(X_\lambda - X_1)}) - (1 - \frac{\lambda}{r})re^{-r(X_\lambda - X_1)}\frac{dX_\lambda}{d\lambda} < 0$, which implies that the optimal λ is 0.

$\square$

Proof of Theorem 6.2

Proof. **Step 1:** $X_1 + X_2 > X^*$.

If $X_1 + X_2 > X^*$, the demand for the good with higher quality is full, and the demand for the other good is positive. Assume that $X_1 > X_2$, then $y_1 = 1$ and the firm chooses $a_2 = a^*_{\lambda_2}(X_2)$ so that $y_2 \in [0, 1]$. The firm will choose λ_1 and λ_2 to satisfy $a_1 = a^*_{\lambda_1}(X_1)$, in order to guarantee the maximum profit. $a^*_{\lambda_1}(X_1) + a^*_{\lambda_2}(X_2) = 1$ implies that $\lambda_1 X_1 + \lambda_2 X_2 = \alpha(X_1 + X_2 - X^*)$. Next, the indifferent condition of the firm implies that $V(X_1, X_2) = (1 - \delta)(A + \lambda_1 X_1 - \gamma + (1 + \lambda_2 X_2)y_2) + \delta V(X_1 + \Delta, X_2 - \Delta) = (1 - \delta)(1 + \lambda_1 X_1 + (A + \lambda_2 X_2)y_2 - \gamma) + \delta V(X_1 - \Delta, X_2 + \Delta)$. Therefore, $\delta(V(X_1 + \Delta, X_2 - \Delta) - V(X_1 - \Delta, X_2 + \Delta)) = \delta(1 - A)(1 - y_2)$. In the limit $\Delta \to 0$, $\frac{\partial V}{\partial X_1} - \frac{\partial V}{\partial X_2} = \frac{r(1-A)(1-y_2)}{2}$ and $rV(X_1) = r(1 + A - \gamma + \lambda_1 X_1 + \lambda_2 X_2) - \frac{1+A+2\lambda_2 X_2}{1-A}(\frac{\partial V}{\partial X_1} - \frac{\partial V}{\partial X_2}) = r(1 + A - \gamma + \alpha(X_1 + X_2 - X^*)) - \frac{1+A+2\lambda_2 X_2}{1-A}(\frac{\partial V}{\partial X_1} - \frac{\partial V}{\partial X_2})$. There is a solution: $V(X_1, X_2) = 1 + A - \gamma + \alpha(X_1 + X_2 - X^*)$ and $y_2(X_1, X_2) = 1$.

Step 2: $X_1 + X_2 \leq X^*$ and $X_1 > X_2$.

In the region $X_1 + X_2 \leq X^*$, the demand for the good with lower quality is zero. Therefore, we need to focus on the quality dynamics of the good with higher quality. Assume that $X_1 > X_2$, then $y_2 = 0$ and the continuation value only depends on X_1, by investing with intensity a_i, the profit from good 1 is $(1 + \lambda_1 X_1)y_1 - a_1(\gamma + (1 - A)y_1)$ and the profit from good 2 is $-\gamma(1 - a_1)$. Since the firm chooses λ_1 first and choose a_1 second, then the firm solves the following problem: $V(X_1) = \max_{\lambda_1} \max_{a_1}(1 - \delta)((1 + \lambda_1 X_1)y_1 - a_1(\gamma + (1 - A)y_1) - \gamma(1 - a_1)) + \delta(a_1 V(X_1 + \Delta) + (1 - a_1)V(X_1 - \Delta))$, where $1 - \delta = r\Delta$.

The existence of an interior solution of a_1 implies that $\delta(V(X_1 + \Delta) - V(X_1 - \Delta)) = (1 - \delta)(1 - A)y_1$. Consequently, $V(X_1) = \max_\lambda(1 - \delta)((A + \lambda X_1)y_1 - \gamma) + \delta V(X_1 + \Delta)$. Define that the optimal λ_1 is λ^*. As $\Delta \to 0$, we have $y_1(X_1) = \frac{2}{r(1-A)}V'(X_1)$, $rV(X_1) =$

$\frac{1+A+2\lambda^* X_1}{1-A} V'(X_1) - r\gamma$. The solution is $V(X_1) = C(1 + \frac{2\lambda^*}{1+A} X_1)^{\frac{r(1-A)}{2\lambda^*}} - \gamma$, $y_1(X_1) = \frac{2C}{1+A}(1 + \frac{2\lambda^*}{1+A} X_1)^{\frac{r(1-A)-2\lambda^*}{2\lambda^*}}$. We show that there is a threshold $X^{**} < X^*$, such that

- If $X_1 \in (X^{**}, X^*)$, the firm strictly prefers to invest in good 1;
- If $X_1 \in (0, X^{**})$, the firm invests with $a_1 = a^*_{\lambda^*}(X_1)$.

If $X_1 \in (X^{**}, X^*)$, the limit condition is $rV(X_1) = \max_{\lambda_1} r(A + \lambda_1 X_1 - \gamma) + V'(X_1) = r(A + \alpha X_1 - \gamma) + V'(X_1)$. Together with a boundary condition $V(X_1) = 1 + A - \gamma + \alpha(X_1 - X^*)$ at $(X^*, 0)$, the solution is that for $X_1 \in (X^{**}, X^*)$, $V(X_1) = 1 + A - \gamma + \alpha(X_1 - X^*) - (1 - \frac{\alpha}{r} - \alpha X^*)(1 - e^{-r(X^* - X_1)})$. A necessary condition is $1 - \frac{\alpha}{r} - \alpha X^* > 0$.

At $X_1 = X^{**}$, we have $y_1(X_1) = 1$ and $V(X_1) = \frac{1+A}{2} + \lambda^* X_1$ and consequently the optimal λ^* is α. To pin down the constant C and X^{**}, we need to match $V(X^{**}-) = V(X^{**}+)$ and $y_1(X^{**}) = 1$. The solution is $X^{**} = X^* - \frac{1}{r} \log \frac{1 - \frac{\alpha}{r} - \alpha X^*}{\frac{1-A}{2} - \frac{\alpha}{r}}$. A necessary condition is that $\alpha < \frac{r(1-A)}{2}$. Therefore, if $\alpha \geq \frac{r(1-A)}{2}$, there is not an exploration stage in which the firm is indifferent between two investment choices.

If $X_1 \in (0, X^{**})$, then the firm choose λ^* to maximize $V(X_1) = C(1 + \frac{2\lambda^*}{1+A} X_1)^{\frac{r(1-A)}{2\lambda^*}} - \gamma$, where C is a constant. It is obvious that $\frac{dV(X_1)}{d\lambda^*} < 0$, so $\lambda^* = 0$. Therefore, $y_1(X_1) = \frac{2}{r(1-A)} V'(X_1)$, $rV(X_1) = \frac{1+A}{1-A} V'(X_1) - r\gamma$, which implies that $y_1(X_1) = e^{-r\frac{1-A}{1+A}(X^{**} - X_1)}$.

Last, we discuss the case in which $1 - \frac{\alpha}{r} - \alpha X^* \leq 0$, then there is not a rewarding stage since the firm's value of specializing is higher than the value of diversifying, i.e., $1 + A - \gamma + \alpha(X_1 - X^*) - (1 - \frac{\alpha}{r} - \alpha X^*)(1 - e^{-r(X^* - X_1)}) > 1 + A - \gamma + \alpha(X_1 - X^*)$. In all, there is only a specialization stage: $a_i = 1$ and $y_i = 1$, for $X_i > X_{-i}$.

$\square$

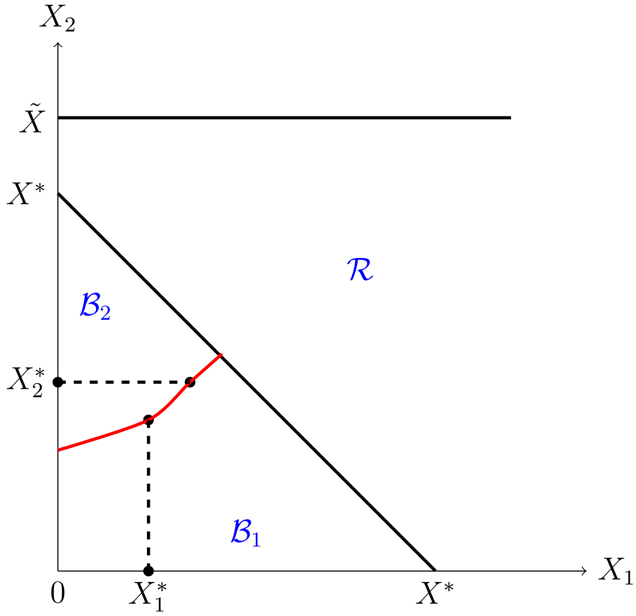

Figure 1: *Quality-Cycle Equilibria*

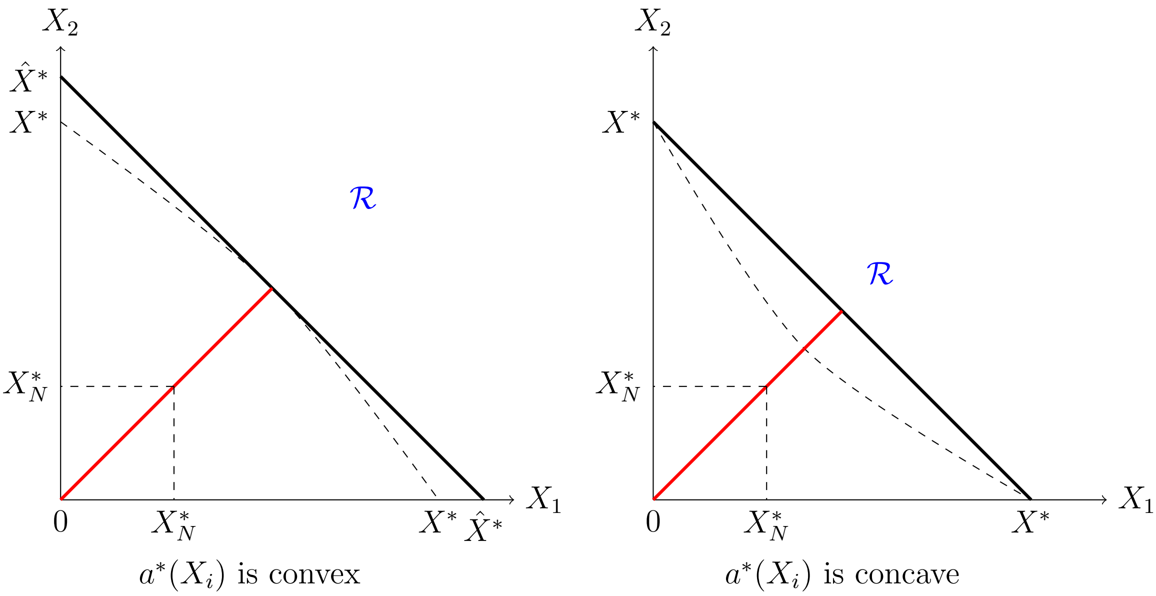

Figure 2: *Non-linearity*