WARS OF ATTRITION WITH PRIVATE RESOURCE CONSTRAINTS

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ABSTRACT. This paper studies wars of attrition in which players have private resource constraints, such as budget limits. We show that whether a player *can* fight longer than her opponent, that is, has a budget advantage, is a key measure of strength in wars of attrition, and uncertainty leads to delay and inefficiency. Furthermore, budget advantage provides a foundation of behavioral types in reputation models: in the unique equilibrium surviving a refinement, high budget players always fight and low budget players fight to develop a reputation of "deep pocket" until they deplete their budgets. We then study a generalization with multiple homogeneous prizes, where we obtain an equivalence result between sequential and simultaneous allocations of prizes through wars of attrition: the winner takes all in both formats and the equilibrium outcomes are the same in the continuous-time limit. Finally, we show that the results extend to the case with a continuum of budget levels.

1. INTRODUCTION

Resource constraints, such as financial capabilities, physical capacities, and time, are important in many real-world war-of-attrition type of strategic environments, where agents expend their limited resources to contend for prizes. For instance, firms may be liquidity constrained in competitions: if they were to engage in a price war they may have to draw from their financial reserves to stay in the market.[1] Students who invest in pre-college human capital to compete for college seats may be restrained by their financial resources (Hickman and Bodoh-Creed (2017)). The importance of financial constraints is also widely recognized in strikes and litigations. During strikes many unions have a "war chest" from which members are paid a proportion of their salaries. The size of these war chests can affect the negotiation process as it affects the amount of time the unions

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[1]Many wars of attritions in business end up being exceedingly costly. In the striking case between the British Satellite Broadcasting and Sky TV, the total costs of the two-year war of attrition were about £1.25 million, which resulted in a merger between the two companies (McAfee (2005)). In another famous example, Best Buy and Circuit City entered a war of attrition in the consumer electronics market, and over more than ten years' price competition Circuit City went bankruptcy and exited the market (McGuigan et al. (2016)).

can strike.[2] Likewise, decisions litigants make in legal proceedings can be motivated by their and their opponents' financial strengths.[3] Besides financial resources, agents may also face private deadlines that limit the amount of time they can compete.[4] In many of the above situations, not only do resource constraints cap the lengths of fights, they are crucial in determining the actual lengths and the winners.[5]

Despite their practical significance, little existing work on wars of attrition has examined the strategic impact of resource constraints.[6] In this paper, we provide a simple framework to analyze wars of attrition with private resource constraints. Specifically, in the baseline setting in Sections 2 and 3, we first examine a stylized war of attrition, where two players, who have either high or low budgets (resources), compete for a single prize with known values. Since fighting incurs a cost, budgets limit the length that players can fight. Inspired by the above examples, the player who is more likely to have a low budget than her opponent can be viewed as an ex ante "weaker" player.[7] In the presence of budget uncertainty, there are at most two kinds of equilibria: either one of the players concedes immediately regardless of her budget, or one of the high-budget bidders behaves as a "behavioral type" player who never concedes unless she depletes her budget, in which case the equilibrium dynamics are reminiscent of the "war-of-attrition" equilibrium in a two-sided reputation model (Kreps and Wilson (1982)). In the first kind of equilibrium, it is possible that the weaker player wins the prize at no cost. This equilibrium exists when the weak player holds optimistic beliefs if the strong player fights and the stronger player's value of the prize is low, so

[2]There is considerable evidence of unions collecting and suggesting they have large war chests before impending wage bargaining. See United Auto Workers war chest, Verizon war chest, NFL, German Auto Workers war chest among others.

[3]Baye et al. (2005) argue that an "undesirable feature of the British system is that it might make courts a playing field for only the wealthy. Under the British system, the prospect of having to pay the winner's legal expenses might preclude the poor from seeking justice through the court system."

[4]Having a limited war chest also implies that the amount of unions can strike is limited. The size of war chest or how much unions pay out in the event of a strike may be private information, thus unions may have a private deadline to end the strike by. In inter-firm groups, team members face deadlines for their deliverables and may engage in a war of attrition to see who concedes and carries out the task first. The team members can privately know their own productivity which affects how long they can procrastinate before needing to do the task.

[5]For example, there is empirical evidence that highlights the positive correlation between firms' financial constraints and their innovative performances. See Schroth and Szalay (2010), Hall et al. (2016), and the references therein. Wealth difference among applicants is rooted in the US college admission process (Avery et al. (2003)), as both preparation and application are costly for the students. In a study of litigations Yoon (2010) provides evidence that a party's wealth significantly affects the legal outcomes.

[6]The literature on wars of attrition is large and has many applications, including biological competitions (Smith (1974), Riley (1980)), industry shakeouts (Fudenberg and Tirole (1986)), public good problems (Bliss and Nalebuff (1984)), patent races (Leininger (1991)), business standard settings (Farrell and Simcoe (2012)), litigations (Kennan and Wilson (1989)), bilateral bargaining (Chatterjee and Samuelson (1987), Abreu and Gul (2000)), politics (Alesina and Drazen (1991), Fearon (1998), Dekel et al. (2008, 2009)), and wars (Powell (2017)).

[7]With budget constraints, difference in players' values turns out to be of second order importance as a measure of strength or a prediction of the winning probability.

that it is too costly even for a high-budget player to prove that she has a "deep pocket."[8] While there is some evidence that lends support to this immediate concession equilibrium, such as many early settlements that involve minor compensations from the strong to the weak party before litigation, it relies on seemingly implausible off-equilibrium path beliefs by the weaker player. We thus introduce a refinement that generalizes the D1 criterion introduced by Cho and Kreps (1987) and show that only the second kind of equilibrium survives our refinement.[9] We also provide a full equilibrium characterization, where costly fighting occurs as low budget players seek to establish a "reputation" of having a high budget; moreover, the ex ante stronger player is more likely to win than the weaker player. Therefore, our results are consistent with the conventional wisdoms regarding resource constraints in the applications discussed in the beginning.

The main contribution of our analysis is that budget advantage, or in general the ability to last longer than opponents, provides a foundation for the behavioral types in the classical reputation models, which often feature war-of-attrition equilibria when both sides may have behavioral types.[10] More concretely, in an influential paper Abreu and Gul (2000) show that, with behavioral types who insist on a certain amount of surplus, a bilateral bargaining with frequent offers has a war-of-attrition structure, where rational players incur delay costs to establish their "strategic postures." However, the origin of the specific behavioral types is not clear as in many other studies in the reputation literature.[11] In contrast, in our setting, all players are rational and reputation equilibria emerge as a result of the strategic impact of uncertain budget constraints.[12] Our findings thus echo the eloquent argument of Wilson (1985) regarding modeling reputations:

> "Reputations can explain many behaviors – perhaps too many. It is too easy to suppose that there is unobserved state variable called reputation that explains all that happens. The better approach is to develop a well-specified model in which the effects of reputations are delineated, and that circumscribes the observations that an outside observer might make."

[8]Budget uncertainty plays a key role in such an equilibrium: if players' budget levels are common knowledge, we show that whichever player has a budget advantage wins immediately.

[9]Several recent papers apply the D1 criterion for equilibrium selection. See Bagwell (2017) for a study of commitment types in a sequential-move duopoly and Sanktjohanser (2017) for a study of reputational bargaining with endogenous stubborn demands.

[10]See for example Kreps and Wilson (1982) and Atakan and Ekmekci (2013). See also Mailath and Samuelson (2006) for a systematic treatment of reputation models.

[11]Abreu and Sethi (2003) endogenize the behavioral types in bargaining based on evolutionary stability.

[12]In Section 5 we show that our reputation equilibria do not depend on the assumption of binary budget levels.

To further illustrate the distinct role of private budget constraints, in Section 4, we examine an extension of the baseline model with multiple homogeneous prizes, where budget-constrained players compete in either sequential or simultaneous wars of attrition. Competition for multiple prizes is realistic in real-world situations and highlights the differences that our specification of budget constraints brings compared to the usual variations of the model based on private valuations or behavioral types.[13] In the sequential case, prizes are allocated one after another—each via a war of attrition. In the simultaneous case, players compete for all prizes at once and they can choose to fight for any subset of prizes among those they have not given up. Our main finding is an equivalence result between these two formats under our equilibrium refinement: as the time interval between players' moves converges to zero, both players obtain the same expected payoff in both sequential and simultaneous wars of attrition. In addition, a "winner-take-all" result holds in both cases regardless of the length of the time interval. This prediction stands in sharp contrast to the "anything goes" result in repeated games with behavioral types.[14] It also differs from the predictions based on models without budget constraints.

Finally, in Section 5 we generalize the results to the setting with a continuum of budgets, where we consider a continuous-time war of attrition with one prize.[15] Under a monotone hazard rate assumption on the distributions of budgets, we show that there is a unique monotone pure strategy equilibrium in which players with higher budgets concede later. The structure of the equilibrium is again equivalent to that in reputational bargaining with possibly atomic concessions at time zero.

1.1. **Related Literature.** Besides the papers already mentioned, our paper is related to papers on wars of attrition with known budgets (Leininger (1991) and Dekel et al. (2007)). The equilibrium dynamics under private budget constraints overturn many predictions based on the complete information counterparts: delay occurs and the expected length of fight depends primarily on the perceived strength of players in terms of the relative budget advantage.

Deadlines have been discussed in the Abreu and Gul (2000) model from a different perspective. Fanning (2016) studies reputational bargaining with behavioral types and deadlines. In his model,

[13] In many patent races, firms invest in multiple areas. For example, Volkswagen, whose R&D spending in 2013 was $13.5 billion, invested in hybrid vehicles, reduction of CO2 emission, semi-autonomous driving, and electronic vehicles (source: http://fortune.com/2014/11/17/top-10-research-development/). In many lawsuits, parties sue each other based on multiple grounds. For example, in a long legal battle between Blizzard and Bossland from 2011 to 2017, the two companies have filed various lawsuits against each other. (source: https://nowloading.co/p/the-legal-battle-between-blizzard-and-bossland-over-bots-and-cheats/4240236).

[14] Fudenberg and Maskin (1986) show that a folk theorem holds in finitely repeated games with suitable choices of behavioral types.

[15] Section 5 studies the case of symmetric players. The asymmetric case is studied in the online appendix.

a stochastic deadline, common to both players, may arrive before players make a deal, which leads to a loss of surplus to both players. In our case, deadlines are individual and type specific, when a certain type of player reaches her deadline, she has to concede the entire surplus to her opponent. Fanning (2016) offers a reputation-based explanation of the fact that many deals frequently happen right before the deadline. Our results complement his findings by highlighting the strategic aspect of private deadlines.

In addition, our analysis of the *dynamic* wars of attrition differs from the analysis of the *static* second-price all-pay auctions with budget constraints in several ways.[16] Kotowski and Li (2014) study symmetric second-price all-pay auctions with private budgets and affiliated values.[17] They identify sufficient conditions for the existence of a symmetric equilibrium with a particular form, but they do not discuss the possibility of other equilibria. In fact, it is well-known that the classic wars of attrition or second-price all-pay auctions have a continuum of equilibria even when players are symmetric (Hendricks et al. (1988)).[18] In contrast, we focus on incomplete information in budgets, allow for asymmetries in both values and budgets, and give a complete characterization of the weak perfect Bayesian equilibria. Furthermore, the idea of our equilibrium refinement is to rule out implausible beliefs for off-equilibrium path plays, which is line with the literature on equilibrium selection in extensive-form games. Therefore, our analysis suggests that the dynamic aspect of wars of attrition can be constructive to narrow down the equilibrium predictions.

Related to the equilibrium multiplicity issue discussed above, it is also instructive to compare our framework with wars of attrition with other kinds of private information instead of budgets. Fudenberg and Tirole (1986) study a dynamic oligopoly model with entry and exit decisions and private costs of fighting. Ponsati and Sákovics (1995) model uncertainty about private values and allow asymmetry between the players in a war of attrition. Abreu et al. (2015) study a bargaining model with one-sided uncertainty regarding a player's discount factor. All the above papers prove equilibrium uniqueness under the assumption that there is a positive probability that each player may be a commitment type who fights forever. As we discussed in the previous paragraph, the existence of commitment types is essential for equilibrium uniqueness. While the equilibrium

[16]See also Krishna and Morgan (1997) for an early analysis of the static all-pay auctions and wars of attrition with symmetric bidders and no budget limits.

[17]Kotowski and Li (2014) also study first-price all-pay auctions. See also Kotowski and Li (2014).

[18]Multiple equilibria lead to difficulties in revenue comparisons and comparative statics. A common method to obtain uniqueness in the literature is to specify perturbations of the original game. See for example Kornhauser et al. (1989), Myatt (2005) and Kim et al. (2017). Nevertheless, different perturbations can lead to selections of different equilibria. As a comparison, we do not refer to perturbations and show that equilibria in our setting are of at most two kinds.

predictions in their models share similar features as our "reputation" equilibrium, such as delay and comparative statics, a first difference is that there can also be an immediate concession equilibrium in our model with private budgets. More importantly, from an empirical viewpoint, the existence of commitment types is not easy to justify: the model would predict wars last forever with a strictly positive probability, whereas concessions always seem to occur in real world situations. In contrast, such an inconsistency never happens in our setting, as budget advantage serves the role of commitment types. Finally, while players' values are commonly known in our model, introducing private values would purify the mixed strategy equilibrium and thus would not change the equilibrium structure; in this sense, our framework also complements the private-value models.

2. MODEL

We begin by considering a benchmark model that, despite its simplicity, is rich enough to deliver our main message. We study more general models in Sections 4 (multiple prizes) and Sections 5 (continuum of budget levels). Consider the following two-player war of attrition. Players 1 and 2 compete for an indivisible prize. Time is discrete and indexed by $t = 1, 2, \dots, \infty$. For $i = 1, 2$, let $V_i > 0$ denote player i's value of each prize, and let B_i denote player i's budget. Assume V_1 and V_2 are common knowledge, but B_i is player i's private information. In particular, $B_i \in \{B_l, B_h\}$ with $0 < B_l < B_h < \infty$. Suppose that B_1 and B_2 are independently distributed, the prior probability that $B_1 = B_h$ is $\rho_1 \in (0,1)$, and the prior probability that $B_2 = B_h$ is $\eta_1 \in (0,1)$. We will refer to player i with the high budget as ih and player i with the low budget as il.

Players alternate in their moves with player 1 taking actions in odd periods and player 2 in even periods. A player who is called upon to move can "stay" by paying a cost ε, which is sunk and is financed from her budget, or concede. When one player concedes, the game ends and the prize is allocated to the other player.[19] We assume the budgets B_l and B_h are multiplications of ε and $\min\{V_1, V_2\} > 2\varepsilon$.[20] Calling the total sunk cost of a player her "total bid," we assume that players

[19] Note that in the current formulation tie-breaking favors player 2 in the case of ties. This, together with the assumption that players alternate in their moves, introduces a slight asymmetry in the model and the characterization of equilibria. Nevertheless, we choose this formulation because it allows us to generalize the existing equilibrium refinement criteria in one-shot signaling games to the dynamic games considered here. Moreover, in the limit as ε converges to zero, we shall see that the equilibrium outcome converges to that in the continuous-time wars of attrition. Third, if the model is perturbed to allow players to have different supports for their budget levels, then the current tie-breaking rule corresponds to assuming that a high budget player 2 can always outbid player 1. Finally, this formulation is also realistic in some real-world situations and has been widely adopted in the literature on wars of attrition. See Leininger (1991) and Dekel et al. (2007) for examples.

[20] Both assumptions are innocuous since we will focus on the limiting case in which ε is close to zero as an approximation of the standard continuous-time wars of attrition.

cannot bid more than their budgets. With this in place the payoffs are as follows.

$$u_1 = \begin{cases} V_1 - \frac{t+1}{2}\varepsilon; \text{ if player 2 concedes in } t+1 \\ -\frac{t-1}{2}\varepsilon; \text{ if player 1 concedes in } t \end{cases} \qquad \text{for } t = 1,3,5\dots$$

$$u_2 = \begin{cases} V_2 - \frac{t-2}{2}\varepsilon; \text{ if player 1 concedes in } t-1 \\ -\frac{t-2}{2}\varepsilon; \text{ if player 2 concedes in } t \end{cases} \qquad \text{for } t = 2,4,6\dots$$

A pure strategy of a player is to decide whether to stay or to concede as long as she has not conceded the prize and has not depleted her budget. In order to describe a player's mixed strategy, for $k \in \{l,h\}$ and $t = 1,2,\dots,B_h/\varepsilon$, let β^{1k}_{2t-1} denote the probability that type $1k$ player 1 concedes in period $2t-1$ and β^{2k}_{2t} the probability that type $2k$ player 2 concedes in period $2t$, provided that they have not conceded yet. Given a player's belief about other's strategies, let ρ_t and η_t be the updated posteriors that players 1 and 2 have high budgets respectively. The equilibrium concepts we use is weak perfect Bayesian equilibrium (wPBE) which follows the usual definition.[21] In addition, we will propose a refinement of wPBE that is in line with the D1 refinement in signaling games.[22]

3. Analysis

In this section, we first (Subsection 3.1) characterize the wPBE of the game, which are either an immediate concession equilibrium in which player 2, despite having an advantage, concedes immediately or an equilibrium in which player $2h$ never concedes. We then (Subsection 3.2) introduce the equilibrium refinement and show that (Subsection 3.3) only the latter equilibrium survives the refinement.

3.1. **Weak perfect Bayesian equilibria.** To see how budget constraint may affect the equilibrium in wars of attrition, let us briefly discuss the degenerate case in which players' budgets are finite and commonly known. Intuitively, if a player has a larger budget than her opponent, at the time when her opponent is about to run out of budget, the player can guarantee a win by bidding a few times, which yields a positive continuation surplus that dominates the payoff from conceding. Thus it is a dominant strategy for the player with a budget advantage to bid at that time, and hence her opponent's optimal response is to concede. By backward induction, in the unique equilibrium the player with a budget advantage always bids and her opponent concedes immediately, regardless

[21] In our model, it can be shown that weak perfect Bayesian equilibrium is equivalent to sequential equilibrium.

[22] See for example Cho and Kreps (1987).

of their values. That is, when $B_1 > B_2$, player 1 wins the prize with a total bid ε; when $B_1 \leq B_2$, player 2 wins the prize with a total bid zero.[23]

While the logic of the complete information case is straightforward, it suggests that when budgets are private information, players will signal their budgets by costly bidding. In other words, low budget players may bid for a non-trivial amount of money in order to establish a "reputation" for having a high budget so that her opponent may concede sooner. One caveat to such an argument is that, unlike the classic reputation models with behavioral types (c.f. Kreps and Wilson (1982)), players with a high budget are also rational and they may not always bid in equilibrium; intuitively, if it takes too long to obtain a "reputation" for deep pocket, then it may not be worthwhile even for the high budget player to bid in the first place. In the subsequent analysis we show that this is indeed possible in a wPBE.

We first show (Lemma 3.1) that, if the game has not ended before the low budget player 2 ($2l$) depletes her budget, from then on the high budget player 2 ($2h$) will always bid.[24] This follows from the fact that it becomes common knowledge that player 2 has a high budget when $2l$ is no longer able to bid. Let $T = B_l/\varepsilon$.

Lemma 3.1. *In any wPBE, in any period $t \geq 2T+2$, $2h$ always bids and $1h$ concedes immediately. Furthermore, $2h$ bids for sure in period $t = 2T$.*

Lemma 3.1 implies that in any wPBE the game ends before or in period $2T+2$. Next we consider the behavior of $2h$ in the first $2T$ periods. Without further refinements, there may exist a wPBE in which both $2l$ and $2h$ concede at their first opportunity to move, which is supported by player 1's off-equilibrium belief that assigns probability zero to $2h$ conditional on observing a bid from player 2. This is despite player $2h$ having the advantage eventually as suggested by Lemma 3.1. The following example illustrates such a possibility.

Example 3.2. Suppose $V_1 = V_2 = 2\varepsilon + \delta$ with $\delta \in (\alpha\varepsilon, \varepsilon)$, where $\alpha > 0$ satisfies $\alpha + \alpha^2 = 1$, $B_l = 2\varepsilon, B_h \geq 4\varepsilon$, $\rho_1 > \delta/\varepsilon$ and $\eta_1 \in (0,1)$. Consider the strategy profile in which both types of player 2 concede in period 2, both types of player 1 believe that player 2 has low budget with probability one after player 2 bids in period 2, and both types of player 1 always bid till either they exhaust their budgets or the game reaches period 7. We show that this strategy profile is part of a wPBE.

[23] The asymmetry here follows from the asymmetric tie-breaking assumption.

[24] The proofs of this lemma and other results in the main text are in the appendix B.

First consider $2l$. She can defeat $1l$ if the game moves to period 4. However, she will lose to player $1h$. Therefore, her payoff is either $-\varepsilon$ if she bids in period 2 and concedes in 4, or it is $-2\varepsilon+(1-\rho_1)V_2$ if she bids till she exhausts her budget. Given player 1's strategy there is no belief updating of player 1's type. Finally, note that the latter payoff is also negative if $\rho_1 > \delta/\varepsilon$.

Next consider $2h$. She can defeat $1l$ if she bids in period 4, and she can defeat $1h$ if she bids in periods 4 and 6, since if she bids in period 6, $1h$ would know that player 2 has a high budget and hence would concede in period 7. If $2h$ bids in periods 2 and 4 but concedes in 6, then her payoff will be the same as $2l$ who follows the same strategy. Therefore, we only need to consider the case in which $2h$ bids in periods 2, 4 and 6. In this case, her payoff will be $-2\varepsilon+(1-\rho_1)V_2+\rho_1(V_2-\varepsilon)<0$, where the inequality follows from $\rho_1 > \delta/\varepsilon$. Hence, $2h$ prefers to concede in period 2.

Finally, we check whether player 1's strategy is sequentially rational. If the game reaches period 7, then $1h$ knows that she is competing against $2h$ and hence concedes. In period 5, given that $1h$ believes she is playing against $2l$, she will bid for sure. In period 3, $1h$ always bids as no matter what $2l$ does in period 4, $1h$ gets a positive payoff. What remains to check is whether $1l$ will bid for sure in period 3. Given her belief that she is facing $2l$ with probability one, $1l$ will bid for sure if and only if she believes that $2l$ will concede in the next period. Otherwise, she will lose and get a payoff $-\varepsilon$. $2l$ will concede for sure in period 4 if and only if her payoff from bidding $-\varepsilon+(1-\rho_1)V_2$ is less than zero. That is $\rho_1 > 1-\varepsilon/V_2 = (\varepsilon+\delta)/(2\varepsilon+\delta)$. Since we have $\rho_1 > \delta/\varepsilon$, and $\delta/\varepsilon > (\varepsilon+\delta)/(2\varepsilon+\delta)$ if and only if $\varepsilon\delta+\delta^2 > \varepsilon^2$, so given $1l$'s belief the definition of α, $1l$ will bid for sure in period 3. Finally, we already established that given player 1's beliefs, both $2l$ and $2h$ concede in period 2. Hence, both $1l$ and $1h$ will bid for sure in period 1.
$\square$

In the above example, player 1 holds an optimistic belief (that her opponent has low budget) after the off-equilibrium history in which player 2 bids in period 2. As a result, unless $2l$ depletes her budget it is impossible for $2h$ to prove that she has a high budget. When player 2's value V_2 is smaller than B_h and she believes that player 1 is sufficient likely to have a high budget, it is possible that the cost (by bidding) for $2h$ to convince player 1 that her budget is high is larger than her value of the prize. Therefore, both types of player 2 prefer to concede early.

Four remarks can be drawn from Example 3.2. (1) This equilibrium is independent of η_1. That is, even if player 1 believes that player 2 is very likely to have a large budget, she still bids as she believes that player 2 will concede in the next period for sure; in particular, she believes that only $2l$ could bid in the next period. Her belief is justified because from $2h$'s view point, the value of the prize is not very large and it is too costly to convince player 1 that she has a high budget. (2) Note that when $V_2 > B_h$, even if player 1 holds an optimistic belief, $2h$ has an incentive to wait till player 1 finds out her type; in fact, it is a dominant strategy for $2h$ to bid at every move. Consequently, there is no wPBE in which player 2 concedes for sure in any period if $V_2 > B_h$. (3) In any wPBE if there is some period in which player 2 concedes for sure regardless of her types then, by backward induction, she will concede for sure in period 2; the same reasoning applies to player 1 as well. (4) In the constructed wPBE player 1's optimistic off-equilibrium path belief may not be "reasonable" given that it is more likely for $2h$ to benefit from bidding in period 2 than for $2l$, simply because whenever $2l$ can win the prize $2h$ can also win. This observation motivates our equilibrium refinement to be proposed after the characterization of wPBE, which unifies the predictions regardless of the sizes of V_2 and B_h.

Motivated by Lemma 3.1 one may guess correctly that there will exist an equilibrium in which player $2h$ always bids since she will eventually have an advantage over player 1, as long as player 1 does not have overly optimistic beliefs and her valuation is not too low. Furthermore, as is implied by the above conditional statement and Example 3.2, there may exist equilibria in which player 2 concedes immediately. Thus a question to ask is whether there are any other types of equilibria? Our first main result stated in the next proposition answers this question in the negative. It gives a partial characterization of wPBE based on the above remarks.

Proposition 3.3. *If $V_2 > B_h$, then in any wPBE, $2h$ always bids in every period $2t$, for $t = 1, \ldots, B_h/\varepsilon$. If $V_2 < B_h$, then in any wPBE, on the equilibrium path either both types of player 2 concedes in period 2, or $2h$ always bids in every period $2t$.*

If the player 2's valuation is higher than her budget then she will always bid as she eventually has an advantage and will win for sure and get a positive payoff irrespective of how much she bids. On the other hand if her valuation is below her budget, the proof shows that on the equilibrium path $2h$ either always bids or concedes in the period 2. Since $2l$'s payoff is weakly smaller than $2h$, this means that whenever $2h$ concedes so does $2l$.

3.2. **Equilibrium refinement.** As we mentioned in the remarks above, to support an equilibrium where player 2 concedes immediately, player 1 needs to have "unreasonably" optimistic beliefs after the off-equilibrium path bidding by player 2. To rule out such equilibria that rely on implausible beliefs, we propose a refinement that is a generalization of the D1 criterion for one-shot signaling games proposed in Cho and Kreps (1987), and show that in any wPBE that survives this refinement, $2h$ always bids in every period.

To introduce the idea of our refinement, recall that in the canonical signaling games in which a sender first sends a signal, and a receiver chooses an action after learning the signal, D1 requires that the receiver puts *zero* weight on a type θ sender when an off-equilibrium path signal m is sent, if there is another type θ' sender such that θ' always strictly benefits from deviating to m whenever θ benefits from this deviation but not the other way round. Since in our setting with alternating moves, player 2 can bid or concede in every other period provided that she has not conceded yet, we will consider iterative applications of D1 refinement of wPBE at every node before $2l$ depletes her budget.

Formally, for $t \geq 2$ let η_t denote the posterior probability that player 1 assigns to $2h$ in the beginning of period t. In the last relevant period of the game, period $2T+3$, $1h$ believes that with probablity one bidder 2 has a high budget (i.e., $\eta_{2T+3} = 1$) and hence the continuation game has a unique wPBE (that satisfies D1), in which $1h$ concedes.[25] Recognize this equilibrium continuation strategy as a singleton set $DE_{2T+3}(\eta_{2T+3})$. For any period $t \leq 2T+1$ and any $\rho_t, \eta_t \in [0,1]$, let $DE_t(\rho_t, \eta_t)$ be the set of wPBE strategies that satisfy D1 in the continuation game from period t when the equilibrium strategies from period $t+1$ onward are in $DE_{t+1}(\rho_{t+1}, \eta_{t+1})$ for any feasible beliefs ρ_{t+1} and η_{t+1}.[26]

Definition 3.4. A strategy profile σ is part of a wPBE that satisfies **generalized D1 (GD1)** if $\sigma \in DE_1(\rho_1, \eta_1)$.

Now we show that player 2 will never concede immediately in any wPBE that satisfies GD1. The intuition is that the refinement GD1 essentially imposes a monotonicity condition on the concession times with respect to players' budgets: since high budget players can always fight longer than low budget players and the value from winning is the same regardless of the budget levels, after observing an off-path event of fighting, a player should believe that the opponent must have a high budget.

Proposition 3.5. *In any wPBE that satisfies GD1, $2h$ always bids in period $2t$, for all $t \leq B_h/\varepsilon$.*

[25] The same argument applies to periods later than $2T+3$.

[26] Feasibility here means that players' beliefs respect the budget constraints. For example, in any period $t > 2T$ only $1h$ and $2h$ can bid, so we must have $\rho_{t+1} = 1$ and $\eta_{t+2} = 1$.

In conjuction with Proposition 3.3, we know from Proposition 3.5 that GD1 in our game is not very restrictive as it only rules out an "unintuitive" equilibrium outcome, which may occur only when $V_2 < B_h$. Furthermore, $2h$ always bids and hence always wins the prize ex post, which confirms the intuitive idea that a high budget player 2 always has an advantage in the game as whenever her budget is commonly known she will win the prize. It is interesting that in a wPBE that satisfies GD1, $2h$ behaves as if she is a "behavioral type" in the reputation literature who never concedes, even though her strategy is derived from sequential rationality.[27] From now on, we will refer to a wPBE that satisfies GD1 as an *equilibrium*.

3.3. **Characterization.** Now we give a full characterization of the equilibria of the single-prize case. First note that given $2h$ always bids in equilibrium, $2l$ will never concede with probability one in any period before period $2T+2$. To put it differently, $2l$ bids in equilibrium with positive probability in order to establish a reputation of high budget. Turning to player 1, because tie-breaking favors player 2 by assumption, $1h$ may not always bid in equilibrium, especially when she believes that it is very likely that player 2 has a high budget. If this is the case, then the incentive for $1l$ to build a reputation of high budget is weaker than the incentive for $2l$. On the other hand, if player 1 has a more optimistic belief of facing a low budget opponent, then $1h$ may bid for sure, which in turn gives an incentive for $1l$ to build a reputation of having high budget.

To make the above intuition precise, we first define two finite sequences of threshold beliefs, $\{\rho_t^*\}_{t=1}^{2T}$ and $\{\eta_t^*\}_{t=1}^{2T+1}$, one for each player. Let $\eta_1^* = (1-\varepsilon/V_1)^T$, and for each $t = 1,2,\ldots,T(= B_l/\varepsilon)$, define

$$\rho_{2t-1}^* = \rho_{2t}^* = \left(1-\frac{\varepsilon}{V_2}\right)^{T-t+1},$$

and

$$\eta_{2t}^* = \eta_{2t+1}^* = \left(1-\frac{\varepsilon}{V_1}\right)^{T-t+1}.$$

We assume that the prior beliefs ρ_1 and η_1 satisfy: $\rho_1 \neq \rho_t^*$ and $\eta_1 \neq \eta_t^*$ for all t. Here is an outline of the equilibrium.

(1) From period $2T+2$ onwards, there is no private information about budgets and the game ends immediately with $2h$ winning the prize.

[27] See Kornhauser et al. (1989), Kambe (1999) and Abreu and Gul (2000) for studies of wars of attrition with behavioral types.

(2) In period $2T+1$, $1h$ bids if $-\varepsilon+(1-\eta_{2T+1})V_1>0$, i.e., $\eta_{2T+1}<\eta^*_{2T+1}$; $1h$ concedes if $\eta_{2T+1}>\eta^*_{2T+1}$.

(3) In period $2T$, if $\eta_{2T}>\eta^*_{2T}$ or $\rho_{2T}<\rho^*_{2T}$, then $2l$ will bid for sure; if $\eta_{2T}<\eta^*_{2T}$ and $\rho_{2T}>\rho^*_{2T}$, then $2l$ will mix between conceding and bidding. For $2l$ to be indifferent, the posterior belief after bidding, η_{2T+1}, must jump to η^*_{2T+1} by part (2), since $1h$ must mix in period $2T+1$.

(4) In period $2T-1$, the analysis is a bit involved since we need to solve for both $1h$ and $1l$'s best reply given the posterior beliefs.

(4.1) If $\eta_{2T-1}>\eta^*_{2T-1}$, then both $1h$ and $1l$ concede. To see this, consider the most favorable case for player 1 with $\rho_{2T-1}=1(>\rho^*_{2T-1})$ and $\eta_{2T-1}=\eta_{2T}<\eta^*_{2T}$. Since $2h$ bids and $2l$ mixes in period $2T$, $1h$'s payoff from bidding in $2T-1$ is

$$-\varepsilon+\eta_{2T-1}\cdot 0+(1-\eta_{2T-1})\left(\beta^{2l}_{2T}V_1+(1-\beta^{2l}_{2T})\cdot 0\right),$$

where we use the fact that $2h$ is indifferent between bidding and conceding in period $2T$. From part (3), β^{2l}_{2T} is given by

$$\frac{\eta_{2T-1}}{\eta_{2T-1}+(1-\eta_{2T-1})(1-\beta^{2l}_{2T})}=\eta^*_{2T+1},$$

thus, $1h$'s payoff from bidding $-\varepsilon+(1-\eta_{2T-1})\beta^{2l}_{2T}V_1$ is negative. Intuitively, player 1 can only gain from bidding when $2l$ concedes in the next period. If the probability that this occurs is small enough, then the expected benefit cannot recover the cost of bidding.

(4.2) If $\eta_{2T-1}<\eta^*_{2T-1}$ and $\rho_{2T-1}>\rho^*_{2T-1}$, then both $1h$ and $1l$ bid. This follows from the fact that $2l$ will randomize in period $2T$ and now both types of player 1's payoff from bidding $-\varepsilon+(1-\eta_{2T-1})\beta^{2l}_{2T}V_1$ is positive.

(4.3) If $\eta_{2T-1}<\eta^*_{2T-1}$ and $\rho_{2T-1}<\rho^*_{2T-1}$, then $1h$ bids and $1l$ randomizes between bidding and conceding. To see this, note that the posterior belief ρ_{2T} must jump to ρ^*_{2T} if player 1 bids in period $2T-1$. Moreover, $2l$ will randomize in period $2T$ such that $1l$ is indifferent in period $2T-1$, since otherwise either both $1h$ and $1l$ strictly prefer to bid, which implies $\rho_{2T}=\rho_{2T-1}$, or $1l$ strictly prefers to concede, which implies $\rho_{2T}=1$, in both cases we will reach contradictions.

(5) The analysis of players' best replies in all earlier periods follows the patterns identified in parts (3) and (4). The details are relegated to the appendix. Here we just point out two additional subtleties. First, even though $2h$ always bids as a "behavioral type," the

analysis is different from the usual two-sided reputation argument (c.f. Kreps and Wilson (1982) and Abreu and Gul (2000)), as $1h$ may not always bid in equilibrium. Indeed, under the assumption that prior probabilities are different from the thresholds, it is not possible that both players' posterior beliefs stay at the thresholds along the path in which both players always bid. Instead, we show that depending on the prior beliefs, one and only one player's posterior beliefs will eventually jump to the thresholds, whereas the other player's posterior beliefs always stay either above or below the thresholds.[28] Relatedly, there could be multiplicity of equilibria in some cases, due to the indeterminacy of the randomization probabilities of $1h$ and $1l$. Nevertheless, the equilibrium payoff is always unique for generic prior probabilities.

The following proposition summarizes the above analysis and gives a full equilibrium characterization.

Proposition 3.6. *In any equilibrium, $2h$ always bids. As a consequence, $2l$ never concedes with probability one in period $2t$, for any $t = 1, \dots, T(= B_l/\varepsilon)$. Furthermore, in period $2t-1$, player 1 with any budget level concedes with probability one if $\eta_{2t-1} > \eta^*_{2t-1}$ and in period $2t$, $2l$ bids with probability one if $\eta_{2t-1} > \eta^*_{2t-1}$. In all other cases, some or all types randomize. The equilibrium path takes one of three forms depending on the initial beliefs:*

(1) If $\eta_1 > \eta^*_1 = \eta^*_3 = (1-\varepsilon/V_1)^T$, then both $1h$ and $1l$ concede immediately and player 2 wins the prize at a price zero.

(2) If $\eta_1 < \eta^*_1$ and $\rho_1 < \rho^*_1 = (1-\varepsilon/V_2)^T$, then $1h$ bids for sure in any period $2t-1$ where $t = 1, \dots, T+1$ and concedes for sure in all subsequent periods. Additionally, as long as the game has not ended, $1l$ and $2l$ randomize such that β^{1l}_{2t-1} and β^{2l}_{2t} satisfy

$$\frac{\rho_{2t-2}}{\rho_{2t-2} + (1-\rho_{2t-2})\left(1-\beta^{1l}_{2t-1}\right)} = \rho^*_{2t}, \quad (1-\eta_{2t-1})\,\beta^{2l}_{2t} = \frac{\varepsilon}{V_1}.$$

(3) If $\eta_1 < \eta^*_1$ and $\rho_1 > \rho^*_1$, then $1h$ and $1l$ bid for sure in period 1. For $t \geq 2$, as long as the game continues, $2l$ randomizes in period $2t$ such that β^{2l}_{2t} satisfies $\eta_{2t+1} = \eta^*_{2t+1}$; both $1h$ and $1l$ are indifferent between bidding and conceding in period $2t-1$, and they randomize such that β^{1l}_{2t-1} and β^{1h}_{2t-1} satisfy $\rho_{2t} \geq \rho^*_{2t}$ and

$$\rho_{2t-2}\beta^{1h}_{2t-1} + (1-\rho_{2t-2})\,\beta^{1l}_{2t-1} = \frac{\varepsilon}{V_2}.$$

[28] Note that this does not necessarily occur in the first two periods due to the asymmetric tie-breaking assumption.

Note that while the equilibrium strategies, in case (2) and (3), are not unique for some players, the equilibrium path and payoffs are unique. Thus, as intimated in the introduction, in equilibrium player $2h$ behaves as the behavioral type who fights till she runs out of budget and $2l$ tries to build reputation of being $2h$. Moreover, when the prior belief that player 2 has a high budget is small both low budget types fight to build a reputation as the high budget types always fight.

4. AN EXTENSION TO MULTIPLE PRIZES

In this section, we study an extension of the model in Section 2 where players compete for $N \geq 2$ homogeneous prizes, either sequentially or simultaneously. We assume the budgets B_l and B_h are multiplications of $N\varepsilon$ and $\min\{V_1, V_2\} > 2\varepsilon$. Specifically, we study two different formats: sequential and simultaneous wars of attrition. In sequential wars of attrition, players compete for one prize at a time until all the prizes are allocated.[29] In simultaneous wars of attrition, players compete for all prizes at once; in particular, in any period a player who is called upon to move can bid for any subset of prizes among those she has not conceded. If player i wins $n \in \{0, \ldots, N\}$ prizes with a total bid of b_i then her payoff is $nV_i - b_i$. Finally, we again assume that players' total bids cannot exceed their budgets.

In the sequential wars of attrition, a (pure) behavioral strategy of a player is to decide whether to bid or to concede for each prize being allocated as long as she has not conceded this prize and has not depleted her budget. In the simultaneous wars of attrition, a (pure) behavioral strategy of a player is to bid for a subset (which could be the empty set, meaning that she concedes all prizes) of prizes among those prizes she has not conceded. In both cases, if a player runs out of budget in some period, she can only concede from then on. As in the single-prize case, we will focus on wPBE that satisfy GD1. By essentially the same argument, in both formats $2h$ never concedes any prize in any equilibrium.

4.1. **Sequential wars of attrition.**

At the beginning of the war of attrition of the n-th prize, the relevant state variables are the posterior beliefs, $\rho_1[n]$ and $\eta_1[n]$, and the remaining (private) budgets, which can equivalently be viewed as how long a player can bid for the prize. Note that regardless of the outcomes of the first $n-1$ prizes, for the same budget type, player 2's budget is always weakly larger that player 1's budget. As a result, we can view allocation of the n-th the prize as a single-prize war of attrition augmented by an endogenous outside option that is derived from the continued wars of attrition of the remaining prizes. Note that unlike the single-prize case

[29] Since prizes are homogeneous by assumption, the exact sequence of prizes does not matter in the model.

analyzed above, there is be an additional asymmetry in the n-th war of attrition: when player 1 won $k(\geq 1)$ of the first $n-1$ prizes, for the same budget type, her budget is $k\varepsilon$ less than that of player 2. While such an asymmetry complicates the analysis, we show that the equilibrium outcomes are still easy to describe and share many features as in the single-prize case.[30]

Formally, we define two sequences of threshold beliefs for the n-th war of attrition as follows: for each $n = 1, 2, \ldots, N$, let $B_{1l}[n]$ denote the low budget type player 1's budget in the beginning of the n-th war of attrition and $T[n] = B_{1l}[n]/\varepsilon$, and define

$$\rho^*_{2t-1}[n] = \rho^*_{2t}[n] = 1 - \frac{\varepsilon}{V_2},$$

for $t = T[n]+n-N, \ldots, T[n]$, and

$$\rho^*_{2t-1}[n] = \rho^*_{2t}[n] = \left(1 - \frac{\varepsilon}{(N+1-n)V_2}\right)^{T[n]+n-N-t}\left(1 - \frac{\varepsilon}{V_2}\right),$$

for $t = 1, \ldots, T[n]+n-N-1$. In addition, for $t = 1, \ldots, T[n]$, define

$$\eta^*_{2t}[n] = \eta^*_{2t+1}[n] = \left(1 - \frac{\varepsilon}{V_1 + (N-n)(V_1 - \varepsilon)}\right)^{T[n]-t+1}.$$

Finally, let $\eta^*_1[n] = \eta^*_2[n]$.

Proposition 4.1. *In any equilibrium, the equilibrium path of the n-th war of attrition takes one of three forms depending on the posterior beliefs at the beginning of this war of attrition:*

(1) If $\eta_1[n] > \eta^*_1[n] = \eta^*_3[n]$, then both $1h$ and $1l$ concede immediately and player 2 wins the n-th prize at a price zero.

(2) If $\eta_1[n] < \eta^*_1[n]$ and $\rho_1[n] < \rho^*_1[n]$, then $1h$ bids for sure in any period, $1l$ randomizes such that $\rho_{2t}[n] = \rho^*_{2t}[n]$, and $2l$ randomizes such that $1l$ is indifferent between conceding and bidding in the previous period.

(3) If $\eta_1[n] < \eta^*_1[n]$ and $\rho_1[n] > \rho^*_1[n]$, then $1h$ and $1l$ bid for sure in period 1. From period 2 onwards, $2l$ randomizes such that $\eta_{2t+1}[n] = \eta^*_{2t+1}[n]$; both $1h$ and $1l$ are indifferent between bidding and conceding, and they randomize such that $\rho_{2t}[n] \geq \rho^*_{2t}[n]$ and $2l$ is indifferent between bidding and conceding in the previous period.

[30] Specifically, since we focus on wPBE that satisfy GD1, whenever player 1 wins a prize, it is common knowledge that player 2 has a low budget. Thus, the continuation game is an asymmetric wars of attrition with one-sided private budgets. We give a complete characterization of equilibria of this game in the appendix.

Notice that for any $\tau = 1, \ldots, 2T+1$ and $n > n'$, the thresholds satisfy $\rho^*_\tau[n] \geq \rho^*_\tau[n']$ and $\eta^*_\tau[n] \geq \eta^*_\tau[n']$; moreover, both inequalities are strict for $\tau < 2(T+n'-N)-1$. Intuitively, as in the single-prize case, the thresholds for a given prize can be thought as the minimal beliefs that guarantee a player an "advantage" in the war of attrition for this prize.[31] When a player has an advantage, she will bid for sure regardless of her budget, and the other "disadvantageous" player will concede with positive probability. The above monotonicity property of the thresholds implies that when a player has an advantage for the n-th prize, then in equilibrium she will have advantage for all the remaining prizes. As a result, in equilibrium if a player is willing to concede the n-th prize, then it is optimal for her to concede all the subsequent prizes. The result is summarized in the following corollary.

Corollary 4.2. *In any equilibrium of the sequential wars of attrition, if a player concedes the n-th prize, then in the continuation game she will concede for all the remaining prizes immediately.*

4.2. **Simultaneous wars of attrition.** When a low budget player bids for n prizes, her budget depletes at a rate that is n times higher than the single-prize case. Thus, if a player never concedes for a strict subset of prizes, the period in which both low budget players run out of budget is $T' = B_l/(N\varepsilon)$. Therefore, the relevant sequences of threshold beliefs are $\{\bar{\rho}^*_t\}_{t=1}^{2T'}$ and $\{\bar{\eta}^*_t\}_{t=1}^{2T'+1}$, where $\bar{\eta}^*_1 = (1-\varepsilon/V_1)^{T'}$,

$$\bar{\rho}^*_{2t-1} = \bar{\rho}^*_{2t} = \left(1 - \frac{\varepsilon}{V_2}\right)^{T'-t+1},$$

and

$$\bar{\eta}^*_{2t} = \bar{\eta}^*_{2t+1} = \left(1 - \frac{\varepsilon}{V_1}\right)^{T'-t+1},$$

for $t = 1, 2, \ldots, T'$.

Proposition 4.3. *In any equilibrium, $2h$ always bids for all prizes. The equilibrium path of the simultaneous wars of attrition takes one of three forms depending on the prior beliefs:*

(1) *If $\eta_1 > \bar{\eta}^*_1$, then both $1h$ and $1l$ concede all prizes immediately and player 2 wins all prizes at a total price zero.*
(2) *If $\eta_1 < \bar{\eta}^*_1$ and $\rho_1 < \bar{\rho}^*_1$, then $1h$ bids for all prizes for sure in any period, $1l$ randomizes such that $\rho_{2t} = \bar{\rho}^*_{2t}$, and $2l$ randomizes such that $1l$ is indifferent between conceding and bidding for all prizes in the previous period.*

[31] Strictly speaking, because of the asymmetric tie-breaking assumption, player 2 will have an advantage whenever the posterior belief about her type is above the threshold regardless of the posterior belief about player 1's type.

(3) If $\eta_1 < \bar{\eta}_1^*$ and $\rho_1 > \bar{\rho}_1^*$, then $1h$ and $1l$ bid for all prizes in period 1. From period 2, $2l$ randomizes such that $\eta_{2t+1} = \bar{\eta}_{2t+1}^*$; both $1h$ and $1l$ are indifferent between conceding and bidding for all prizes, and they randomize such that $\rho_{2t} \geq \bar{\rho}_{2t}^*$ and $2l$ is indifferent between conceding and bidding for all prizes in the previous period.

The intuition of the above result comes from the fact that when a player bids for only a subset of prizes instead of bidding for all, the opponent will be able to compete longer, which implies that the threshold beliefs for the opponent to concede are lower, as a result the opponent has a stronger incentive to bid for the remaining subset of prizes.[32]

4.3. **An equivalence result.** We have shown that in both sequential and simultaneous wars of attrition the winner takes all the prizes. For a fixed bidding cost ε, there is no clear comparison of the equilibrium payoffs in these two formats. However, if we let the length pf period shrink to zero and fix the cost of bidding per unit of time at $c > 0$, i.e., $\varepsilon = c\Delta$, then in the continuous-time limit as Δ converges to zero, players' expected payoffs in any equilibrium are the same in both formats for any generic prior beliefs. To see this, consider the limiting thresholds for the prior beliefs defined in the previous two sections:

$$\lim_{\varepsilon \to 0} \rho_1^*[1] = \lim_{\varepsilon \to 0} \left(1 - \frac{\varepsilon}{NV_2}\right)^{\frac{B_l}{\varepsilon} - N} \left(1 - \frac{\varepsilon}{V_2}\right) = e^{-\frac{B_l}{NV_2}} = \lim_{\varepsilon \to 0} \bar{\rho}_1^*,$$

and

$$\lim_{\varepsilon \to 0} \eta_1^*[1] = \lim_{\varepsilon \to 0} \left(1 - \frac{\varepsilon}{NV_1 - (N-1)\varepsilon}\right)^{\frac{B_l}{\varepsilon}} = e^{-\frac{B_l}{NV_1}} = \lim_{\varepsilon \to 0} \bar{\eta}_1^*.$$

Since the threshold prior beliefs determine players' equilibrium payoffs in both formats, we have the following equivalence result.

Theorem 4.4. *For generic prior beliefs, in the continuous-time limit, a player's equilibrium payoffs are the same whether the prizes are allocated sequentially or simultaneously. Furthermore, in both sequential and simultaneous wars of attrition, in equilibrium if a player wins one prize, she will win all other prizes.*

[32] This intuition is rather different from the logic based on behavioral types and reputation.

5. A Continuum of Private Budgets

In this section, we extend the analysis to the case with a continuum of private budgets. We restrict to the single-prize war of attrition and assume that both players' budgets are independently drawn from the same atomless distribution. For simplicity, we study a symmetric continuous-time war of attrition where the flow cost of bidding is $c > 0$ to both players and each player's value of the prize is $V > 0$.[33] We will focus on symmetric pure-strategy equilibria that are non-decreasing in budget levels.[34]

Specifically, suppose for each $i = 1,2$, player i's private budget B_i is drawn independently from an interval $[\underline{B}, \bar{B}] \subset \mathbb{R}_+$ according to a symmetric probability distribution F with a strictly positive density function $f > 0$. We assume that F satisfies the monotone hazard rate condition, that is, the hazard rate function

$$\lambda(B) \equiv \frac{f(B)}{1 - F(B)}$$

is increasing in B. The next result shows that there is a unique symmetric pure-strategy and monotone equilibrium that is characterized by a stopping time and a cutoff budget level such that a player waits until she depletes her budget if and only if her budget is larger than the cutoff.

Proposition 5.1. *Assume $\lambda(B)$ is increasing. Then there is a unique symmetric pure strategy monotone equilibrium $\tau : [\underline{B}, \bar{B}] \to \mathbb{R}_+$, such that a player with budget $B \in [\underline{B}, \bar{B}]$ concedes at time $\tau(B)$, and τ is given by:*

- *if $B < B^*$, then $\tau(B) = \frac{B^*}{c} - \frac{V}{c} \int_B^{B^*} \lambda(b) db$,*
- *if $B \geq B^*$, then $\tau(B) = \frac{B}{c}$,*

where $B^* \in [\underline{B}, \bar{B}]$ is the unique cutoff that satisfies $B^* = V \int_{\underline{B}}^{B^*} \lambda(b) db$.

Recall that in the binary budget case, the low budget players concede with positive probability in each round. Thus, with positive probability she never depletes her budget. The above equilibrium in the continuum case bears a similar feature. All players with budget levels below B^* concede at a

[33] The result extends to the asymmetric war of attrition. See the online appendix for details.

[34] We will show that the equilibrium is strictly increasing in budget levels, which implies that tie-breaking rules do not matter here. Note that it is non-trivial to prove the existence of equilibria that are monotone in budget, since players' values are complete information and there is no natural single-crossing property for budgets. The motivation for monotonicity is based on the feature that a player with a higher budget can wait longer than a player with a lower budget, as in the binary case.

point strictly before they run out of resources; furthermore, all such players are indifferent between conceding at any time before they run out of budgets.[35]

6. Concluding remarks

In this paper we study wars of attrition where players have private financial constraints. This natural extension of the classic wars of attrition is motivated by many real-world examples where resource constraints affect players' ability to compete. We show that budget constraints generate new predictions, such as the "winner-take-all" result. Furthermore, given that the war-of-attrition structure is common in two-sided reputation models, our analysis suggests that budget advantage can be viewed as a micro-foundation for certain irrational types.

There are several variations of the benchmark model that are of independent interests. In this paper, we assume that both players in the wars of attrition have hard budget limits or deadlines and that the prizes are exogenously given. We leave the question of how other forms of financial capabilities may affect the equilibrium predictions for future research. From a design perspective, it is also interesting to endogenize the prizes and players' budget constraints. Relatedly, our "winner-take-all" result applies when prizes are homogeneous, it is interesting to see whether the result generalizes to the heterogeneous prizes case. Another open question is to extend the model to more than two players.[36] Finally, depending on the specific applications, it might be fruitful to extend the analysis to settings where players either have time-varying budgets or can choose how much cost to incur at every move.[37]

Appendix A. One-sided private budgets

In this section, we analyze a one-sided private budget setting where player 2's budget B_2 is commonly known to both players and player 1's budget B_1 is her private information, which takes one of two values, B_l or B_h, with $B_l \leq B_2 < B_h$. The prior probability that $B_1 = B_h$ is $\rho_1 \in (0,1)$. For $t \geq 2$, let ρ_t denote player 2's posterior belief that player 1 has a high budget in the beginning of period t. Let $T = B_l/\varepsilon$. We first consider the single-prize war of attrition.

[35] This is in contrast to the monotone pure-strategy equilibria in private value models where players have strict incentives to concede at certain times depending on their values.

[36] See Bulow and Klemperer (1999) for an elegant study of a $(N+K)$-player and N-prize generalized war of attrition without budget constraints where each player has a unit demand.

[37] Dekel et al. (2007) analyze the latter extension under the assumption that players' budget are complete information. Hörner and Sahuguet (2011) study a related setting with binary private values but without budget constraints.

As an analogy to the two-sided private budgets case, we first show that $1h$ always bids in any wPBE that satisfies GD1. The proofs are in the online appendix.

Lemma A.1. *In any wPBE that satisfies GD1, $1h$ always bids in period $2t-1$, for all $t = 1, \ldots, B_h/\varepsilon$.*

We then show that there is a unique wPBE that satisfies GD1 and provide a complete characterization of the equilibrium. Define the sequence of threshold beliefs $\{\rho_t^*\}_{t=1}^{2T}$ as

$$\rho_{2t-1}^* = \rho_{2t}^* = \left(1 - \frac{\varepsilon}{V_2}\right)^{T-t+1},$$

for each $t = 1, \ldots, T$.

Proposition A.2. *There exists a unique wPBE that satisfies GD1. In equilibrium, $1h$ always bids. For each $t = 1, \ldots, T (= B_l/\varepsilon)$, in period $2t-1$, $1l$ bids with probability one if $\rho_{2t-1} \geq \rho_{2t}^*$ and concedes with positive probability if $\rho_{2t-1} < \rho_{2t}^*$; in period $2t$, player 2 concedes if $\rho_{2t} > \rho_{2t}^*$ and bids if $\rho_{2t} < \rho_{2t}^*$.*

Moreover, on the equilibrium path, if $\rho_1 > \rho_1^* = (1 - \varepsilon/V_2)^T$, then $1l$ bids for sure and player 2 concedes; if $\rho_1 < \rho_1^* = (1 - \varepsilon/V_2)^T$, then $1l$ concedes with probability β_{2t-1}^{1l}, which satisfies

$$\frac{\rho_{2t-1}}{\rho_{2t-1} + (1 - \rho_{2t-1})(1 - \beta_{2t-1}^{1l})} = \rho_{2t}^*,$$

and player 2 concedes with probability $\beta_{2t}^2 = \varepsilon/V_1$.

Next we consider the case with multiple homogeneous prizes as in the two-sided private budgets setting studied in the main text. We characterize the unique equilibrium for sequential and simultaneous wars of attrition, respectively.

Proposition A.3. *There is a unique equilibrium in the sequential wars of attrition. In equilibrium, $1h$ always bids. In the n-th war of attrition, if $\rho_1[n] > \rho_1^*[n]$, then $1l$ bids and player 2 concedes; if $\rho_1[n] < \rho_1^*[n]$, $1l$ concedes with probability $\beta_1^{1l}[n]$ such that*

$$\frac{\rho_1[n]}{\rho_1[n] + (1 - \rho_1[n])(1 - \beta_1^{1l}[n])} = \rho_2^*[n],$$

and player 2 concedes with probability ε/V_1.

Proposition A.4. *There is a unique equilibrium in the simultaneous wars of attrition. In equilibrium, $1h$ always bids for all prizes, if $\rho_1 > \bar{\rho}_1^*$, then $1l$ bids and player 2 concedes; if $\rho_1 < \bar{\rho}_1^*$, then*

$1l$ concedes all prizes with probability $\beta_1^{1l}(N)$ such that

$$\frac{\rho_1}{\rho_1+(1-\rho_1)(1-\beta_1^{1l}(N))}=\bar{\rho}_2^*$$

and bids for all prizes with probability $1-\beta_1^{1l}(N)$, player 2 concedes all prizes with probability ε/V_1 and bids for all prizes with probability $1-\varepsilon/V_1$.

Appendix B. Proofs

Proof of Lemma 3.1. For any $t \geq 2T+2$, only high budget players remain in the game. Since tie-breaking favors player 2, following the above argument for the complete information case, $1h$ concedes in period $t \geq 2T+3$ and $2h$ bids in period $t \geq 2T+2$.

To prove the final part of the lemma, we calculate the continuation payoff U_{2T}^{2h} of $2h$ in period $2T$. Let β_{2T+1}^{1h} be the probability with which $1h$ concedes in $2T+1$. Then,

$$U_{2T}^{2h}=-\varepsilon+(1-\rho_{2T})V_2+\rho_{2T}\left(\beta_{2T+1}^{1h}V_2+\left(1-\beta_{2T+1}^{1h}\right)(V_2-\varepsilon)\right)>0,$$

where the inequality follows from $V_2>2\varepsilon$.

If $V_2>B_h$, then it is a dominant strategy for $2h$ to bid in every period until she depletes her budget. If $V_2<B_h$, suppose by contradiction that $2h$ concedes with positive probability in period $2t$ with $t\geq 2$. It follows that the posterior belief that player 1 has a high budget in period $2t$ must be larger than zero, since otherwise it is common knowledge that player 1 has lower budget than player 2 and following the complete information case player 2 bids for sure. If $V_2>(T-t+1)\varepsilon$, then $2h$ obtains a positive payoff by always bidding from period $2t$ to $2T+2$, a contradiction.

Proof of Proposition 3.3. Suppose $V_2<(T-t+1)\varepsilon$. If $2h$ strictly prefers to concede in period $2t$, so does $2l$; otherwise, $2l$ obtains a non-negative payoff from bidding in $2t$. Since $2h$ can always follows $2l$'s strategy, this means that $2h$ also obtains a non-negative payoff from bidding in $2t$, a contradiction. However, if both $2h$ and $2l$ concede in period $2t$, then player 1, regardless of her type, bids for sure in period $2t-1$. This implies that both $2h$ and $2l$ concede in period $2t-2$. Continuing this argument, it follows that $2h$ and $2l$ concede in period 2. If $2h$ is indifferent between conceding and bidding in period $2t$, then $2l$ must also weakly prefer to concede; otherwise $2h$ can always follow $2l$'s strategy and obtain a strictly positive payoff, a contradiction. In addition, whenever $2h$ bids with positive probability, $2l$ also does so; otherwise, $2l$ can profitably deviate to bidding, after which player 1 will concede for sure. The above argument implies that both $2h$ and $2l$'s continuation payoffs from bidding in period $2t$ are zero. Since from period $2T+2$ onward $2h$

obtains a positive continuation payoff whereas $2l$ obtains zero, on the equilibrium path the game cannot move to period $2T+2$ with positive probability. This means that player 1, regardless of her types, concedes for sure in some period $2t'+1$ where $t' \leq T$. Since both $2l$ and $2h$ can bid in period $2t'$, applying backward induction again, on the equilibrium path both $1l$ and $1h$ will always concede in every period before $2t'+1$. However, this implies that both $2l$ and $2h$ can win for sure if they bid in $2t$, a contradiction. Hence, in any wPBE either both $2h$ and $2l$ concede in period 2 or $2h$ always bids.

Proof of Proposition 3.5. We prove this statement backwards. Without loss of generality, we assume $V_2 < B_h$.[38] By Lemma 3.1, $2h$ always bids from period $2T$ onwards, where $T = B_l/\varepsilon < B_h/\varepsilon$. This implies that in any equilibrium $2l$ will not concede for sure in period $2T$.[39] Consequently, the total probability that player 1 concedes in period $2T+1$, β^{1h}_{2t+1}, is no less than ε/V_2. Note that the continuation payoff of $2h$ from bidding in period $2T$ is strictly positive and is weakly larger than that of $2l$.

In period $2T-2$, suppose $2h$ concedes with positive probability $\beta^{2h}_{2T-2} > 0$. If $\beta^{2h}_{2T-2} < 1$, then $2h$ is indifferent between bidding and conceding. That is,

$$-\varepsilon + \bar{\beta}_{2T-1}V_2 + (1-\bar{\beta}_{2T-1})\left(-\varepsilon + \beta^{1h}_{2T+1}V_2 + (1-\beta^{1h}_{2T+1})(-\varepsilon + V_2)\right) = 0,$$

where $\bar{\beta}_{2T-1} = \rho_{2T-1}\beta^{1h}_{2T-1} + (1-\rho_{2T-1})\beta^{1l}_{2T-1}$, and the left-hand side of the equation is $2h$'s payoff from bidding in period $2T-2$. Since $V_2 > 2\varepsilon$, we have $\bar{\beta}_{2T-1} < 1$ and $\beta^{1h}_{2T+1} < 1$. Then $2l$'s payoff from bidding in period $2T-2$ is

$$-\varepsilon + \bar{\beta}_{2T-1}V_2 + (1-\bar{\beta}_{2T-1})\left(-\varepsilon + \beta^{1h}_{2T+1}V_2 + (1-\beta^{1h}_{2T+1})\cdot 0\right) < 0,$$

which implies that $2l$ should concede in period $2T-2$ in equilibrium. However, if $2l$ deviates to bidding, player 1's posterior belief is $\eta_{2T-1} = 1$; consequently, player 1 concedes for sure in period $2T-1$. Therefore, this is a profitable deviation for $2l$. Hence $\beta^{2h}_{2T-2} \in (0,1)$ cannot be true in any equilibrium.

Suppose $\beta^{2h}_{2T-2} = 1$, that is, $2h$ concedes in period $2T-2$. Then by the same argument as in the previous paragraph, $2l$ must also concede in $2T-2$. Hence, bidding in period $2T-2$ is off the equilibrium path. Note that for any mixed action of player 1 in subsequent periods that can result a strictly positive payoff for $2l$, $2h$ can also obtain a strictly positive payoff, but not vice versa.

[38]If $V_2 > B_h$, then Proposition 3.5 implies that $2h$ always bids in any wPBE, so there is no off-equilibrium history where player 2 concedes for sure.

[39]Otherwise, $2l$ can deviate to bidding, in which case player 1 will concede in the next period.

Therefore, according to GD1, player 1's posterior belief in period $2T-1$ must satisfy $\eta_{2T-1}=1$. Thus, player 1 concedes for sure in period $2T-1$. But then it is profitable for both $2l$ and $2h$ to bid in period $2T-2$. That is, $\beta^{2h}_{2T-2}\neq 1$ and hence, $2h$ bids in period $2T-2$ in any equilibrium. In addition, as in period $2T$, $2l$ will not concede for sure in period $2T-2$, which in turn implies that player 1 concedes with a probability at least ε/V_2 in period $2T-1$. Finally, the continuation payoff of $2h$ from bidding in period $2T-2$ is again strictly positive and weakly larger than that of $2l$.

Using backwards induction, in period $2T-\tau$ for any $\tau=4,6,\ldots,2T-2$, given that the continuation play must be a wPBE that satisfies GD1, by the same argument as in the previous paragraph, if $2h$ does not bid for sure then it must be that both types of bidder 2 concedes for sure; consequently, the continuation payoff of $2h$ is strictly positive and is weakly larger than that of $2l$. Since bidder 1's off-equilibrium path belief after bidder 2 bids in period $2T-\tau$ must satisfy GD1, it must assign probability one to $2h$, in which case both $2l$ and $2h$ would deviate to bidding in period $2T-\tau$. Therefore, $2h$ bids for sure in period $2T-\tau$ in any wPBE that satisfies GD1.

Proof of Proposition 3.6. We proceed backwards. The equilibrium strategies in periods $2T-1$ and onwards are provided in the main text. Here we continue from period $2T-2$. If $\eta_{2T-2}>\eta^*_{2T-2}$, then $2l$ bids since both $1h$ and $1l$ concede in the next period. If $\eta_{2T-2}<\eta^*_{2T-2}$ and $\rho_{2T-2}<\rho^*_{2T-2}$, then $2l$ also bids since $1l$ randomizes in period $2T-1$ such that ρ_{2T} jumps to ρ^*_{2T}. If $\eta_{2T-2}<\eta^*_{2T-2}$ and $\rho_{2T-2}\geq\rho^*_{2T-2}$, then $2l$ randomizes such that $\eta_{2T-1}=\eta^*_{2T-1}$, in which case both $1h$ and $1l$ will randomize in period $2T-1$ such that

$$\rho_{2T-1}\beta^{1h}_{2T-1}+(1-\rho_{2T-1})\beta^{1l}_{2T-1}=\frac{\varepsilon}{V_2} \tag{1}$$

and

$$\frac{\rho_{2T-1}(1-\beta^{1h}_{2T-1})}{\rho_{2T-1}(1-\beta^{1h}_{2T-1})+(1-\rho_{2T-1})(1-\beta^{1l}_{2T-1})}=\frac{\rho_{2T-1}(1-\beta^{1h}_{2T-1})}{1-\frac{\varepsilon}{V_2}}\geq\rho^*_{2T},\text{[40]} \tag{2}$$

where the first equation ensures that $2l$ is willing to randomize in period $2T-2$ and the second inequality implies that $2l$ will randomize in period $2T$, which in turn provides incentives for $1h$ and $1l$ to randomize in period $2T-1$. Note that if $\rho_{2T-1}\in[(1-\varepsilon/V_2)^2,(1-\varepsilon/V_2))$ then one solution to (1) and (2) is $\beta^{1h}_{2T-1}=0$ and $\beta^{1l}_{2T-1}=(\varepsilon/V_2)/(1-\rho_{2T-1})\in(0,1)$, it is also the unique solution when $\rho_{2T-1}=(1-\varepsilon/V_2)^2$; if $\rho_{2T-1}>(1-\varepsilon/V_2)$ then $\beta^{1h}_{2T-1}\in(0,1)$.

Inducting backwards, in period $2t-1$ for $t=1,\ldots,T-1$, we have the following:

[40] Note that $\rho_{2T-1}=\rho_{2T-2}$.

- If $\eta_{2t-1} > \eta^*_{2t-1}$, given that $2l$'s concession probability in the next period is less than ε/V_1 (which is zero if $\eta_{2t-1} > \eta^*_{2t}$ or leads to $\eta_{2t+1} = \eta^*_{2t+1}$ if $\eta_{2t-1} < \eta^*_{2t}$) and that both $1h$ and $1l$ are indifferent between bidding and conceding in period $2t+1$, it is optimal for both $1h$ and $1l$ to concede in period $2t-1$.
- If $\eta_{2t-1} < \eta^*_{2t-1}$ and $\rho_{2t-1} > \rho^*_{2t-1}$, since $2l$ will randomize in period $2t$ such that $\eta_{2t+1} = \eta^*_{2t+1}$, both $1h$ and $1l$ prefer to bid in period $2t-1$.
- If $\eta_{2t-1} < \eta^*_{2t-1}$ and $\rho_{2t-1} < \rho^*_{2t-1}$, then $1l$ has to randomize in period $2t-1$ such that $\rho_{2t} = \rho^*_{2t}$, which requires $2l$ to concede in period $2t$ with probability ε/V_1 to provide incentive for $1l$'s randomization. Moreover, since the updated beliefs about player 2's type satisfy $\eta_{2s+1} < \eta^*_{2s+1}$ for all $s = t, \dots, T$, it follows that $1h$'s payoff from bidding is strictly positive. Therefore, $1h$ bids for sure in period $2t-1$.

In addition, in period $2t$ for $t = 1, \dots, T-2$, we have:

- If $\eta_{2t} > \eta^*_{2t}$, then $2l$ bids, and in the next period both $1h$ and $1l$ concede.
- If $\eta_{2t} < \eta^*_{2t}$ and $\rho_{2t} < \rho^*_{2t}$, then $2l$ bids, and in the next period $1l$ randomizes such that $\rho_{2t+2} = \rho^*_{2t+2}$.
- If $\eta_{2t} < \eta^*_{2t}$ and $\rho_{2t} \geq \rho^*_{2t}$, then $2l$ randomizes such that $\eta_{2t+1} = \eta^*_{2t+1}$, and in the next period both $1h$ and $1l$ randomize such that

  $$\rho_{2t+1}\beta^{1h}_{2t+1} + (1-\rho_{2t+1})\beta^{1l}_{2t+1} = \frac{\varepsilon}{V_2}$$

  and

  $$\frac{\rho_{2t+1}(1-\beta^{1h}_{2t+1})}{\rho_{2t+1}(1-\beta^{1h}_{2t+1}) + (1-\rho_{2t+1})(1-\beta^{1l}_{2t+1})} \geq \rho^*_{2t+2}.$$

Hence, on the equilibrium path, if $\eta_1 > \eta^*_1$, then player 2 wins the prize immediately; if $\eta_1 < \eta^*_1$ and $\rho_1 < \rho^*_1$, then $1h$ always bids, both $1l$ and $2l$ concede with positive probabilities in every period; if $\eta_1 < \eta^*_1$ and $\rho_1 > \rho^*_1$, then $1h$ and $1l$ bid in the first period then randomize in all subsequent periods, and $2l$ always randomizes in every period. □

Proof of Proposition 4.1. We proceed backwards from the N-th war of attrition. In this case the result follows from Proposition 3.6 and the analysis in appendix A.

Suppose the result holds for the $(n+1)$-th war of attrition and consider the n-th war. Let $B_{ik}[n]$ denote type $k \in \{l, h\}$ player i's budget in the beginning of the n-th war of attrition. Since $2h$ never concedes, by the analysis in appendix A, if player 2 concedes any of the first n prizes before $1l$ runs out of budget, then it becomes common knowledge that her budget is low, the result then follows from the analysis in appendix A. Next suppose that player 2 has won all the first $n-1$ prizes. In the n-th war of attrition, we will again start from the period right after $1l$ depletes her budget. Let $T[n] = B_{1l}[n]/\varepsilon$. In period $2T[n]+1$, $1h$ bids if $-\varepsilon + (1-\eta_{2T[n]+1}[n])[V_1 + (N-n)(-\varepsilon + V_1)] > 0$, i.e., $\eta_{2T[n]+1}[n] < \eta^*_{2T[n]+1}[n]$. $1h$ concedes if $\eta_{2T[n]+1}[n] > \eta^*_{2T[n]+1}[n]$.

In period $2T[n]$, if $\eta_{2T[n]}[n] > \eta^*_{2T[n]}[n]$ or $\rho_{2T[n]}[n] < \rho^*_{2T[n]}[n]$, then $2l$ bids, since her payoff from bidding is at least $-\varepsilon + (1-\rho_{2T[n]}[n])(N-n+1)V_2$, and her payoff from conceding is $(1-\rho_{2T[n]}[n])(N-n)V_2$, since $1l$ has run out of budget and player 2 will win all subsequent prizes if player 1's type is $1l$ even if she concedes in period $2T[n]$. If $\eta_{2T[n]}[n] < \eta^*_{2T[n]}[n]$ and $\rho_{2T[n]}[n] > \rho^*_{2T[n]}[n]$, then $2l$ randomizes such that $\eta_{2T[n]+1}[n] = \eta^*_{2T[n]+1}[n]$, as $1h$ will randomize in period $2T[n]+1$.

In period $2T[n]-1$, (1) if $\eta_{2T[n]-1}[n] > \eta^*_{2T[n]-1}[n]$, then player 1's payoff from bidding is $-\varepsilon + (1-\eta_{2T[n]-1}[n])\beta^{2l}_{2T[n]}[n](V_1 + (N-n)(-\varepsilon + V_1))$ if $\eta_{2T[n]-1}[n] < \eta^*_{2T[n]+1}[n]$ and $-\varepsilon$ if $\eta_{2T[n]-1}[n] \geq \eta^*_{2T[n]+1}[n]$ In both cases, the payoff from bidding is negative, so both $1h$ and $1l$ concede; (2) If $\eta_{2T[n]-1}[n] < \eta^*_{2T[n]-1}[n]$ and $\rho_{2T[n]-1}[n] > \rho^*_{2T[n]-1}[n]$, then both $1h$ and $1l$ bid, since the continuation payoff from conceding is zero by the induction hypothesis, and the continuation payoff from bidding is

$$\begin{aligned}
&-\varepsilon + (1-\eta_{2T[n]-1}[n])\beta^{2l}_{2T[n]}(V_1 + (N-n)(-\varepsilon + V_1)) \\
&> -\varepsilon + \frac{\varepsilon}{V_1 + (N-n)(-\varepsilon + V_1)}(V_1 + (N-n)(-\varepsilon + V_1)) > 0.
\end{aligned}$$

(3) If $\eta_{2T[n]-1}[n] < \eta^*_{2T[n]-1}[n]$ and $\rho_{2T[n]-1}[n] < \rho^*_{2T[n]-1}[n]$, then $1h$ bids and $1l$ randomizes, which follows from the induction hypothesis and the fact that in equilibrium the posterior belief satisfies $\rho_{2T[n]}[n] = \rho^*_{2T[n]}[n]$. We note that the argument here is essentially the same as the proof of Proposition 3.6, given the induction hypothesis and the new threshold beliefs. Therefore, continuing this argument, we conclude that the result holds for the n-th war of attrition. □

Proof of Corollary 4.2. Let $B_{ik}[n]$ denote type $k \in \{l,h\}$ player i's budget in the beginning of the n-th war of attrition. First suppose player 1 won some prizes on the equilibrium path before the N-th war of attrition. Consider the first prize that player 1 won, say, prize n. since $2h$ never concedes, it follows that player 2 is known to have a low budget $B_{2l}[n+1]$ from the $(n+1)$-th war of attrition, i.e., $\eta_1[n+1] = 0$. Moreover, $B_{1l}[n+1] < B_{2l}[n+1] < B_{1h}[n+1]$, since whenever player 1 won a prize, her budget is ε less than player 2's for the same type of budget. By the analysis of the one-sided private budget in appendix A, $1h$ never concedes from the $(n+1)$-th war of attrition onwards.

In addition, if $\rho_1[n+1] > \rho_1^*[n+1]$, then $2l$ will concede in all subsequent wars of attritions immediately. Since $2l$ conceded in the n-th war of attrition, it follows that the posterior belief about player 1 in the period that $2l$ conceded is weakly larger than the corresponding threshold belief:

$$\rho_{2(T[n]-B_{1l}[n+1]/\varepsilon)}[n] \geq \rho^*_{2(T[n]-B_{1l}[n+1]/\varepsilon)}[n].$$

Since $\rho_1[n+1] = \rho_{2(T[n]-B_{1l}[n+1]/\varepsilon)}[n]$, it follows from the definition of the thresholds that

$$\rho^*_{2(T[n]-B_{1l}[n+1]/\varepsilon)}[n] > \rho_1^*[n+1].$$

Therefore, $\rho_1[n+1] > \rho_1^*[n+1]$ and $2l$ concedes all subsequent prizes once she conceded in the n-th war of attrition.

Next suppose player 1 did not win any of the first n prizes. Since both $1h$ and $1l$ conceded in the n-th war of attrition, we have

$$\eta_{2(T[n]-B_{1l}[n+1]/\varepsilon)-1}[n] \geq \eta^*_{2(T[n]-B_{1l}[n+1]/\varepsilon)-1}[n].$$

Since $\eta_{2(T[n]-B_{1l}[n+1]/\varepsilon)-1}[n] = \eta_1[n+1]$ and $\eta^*_{2(T[n]-B_{1l}[n+1]/\varepsilon)-1}[n] > \eta_1^*[n+1]$, so $\eta_1[n+1] > \eta_1^*[n+1]$, which implies that player 1 will concede the $(n+1)$-th and all subsequent prizes immediately. $\square$

Proof of Proposition 4.3. The proof is by induction on the number of available prizes n such that in equilibrium both players will either concede or bid for all the prizes. The result is trivially true when $n = 1$.

When $n = 2$, we argue that other than the last two periods when $1l$ is about to deplete her budget, no player will concede only one prize in equilibrium. To see this, suppose both players have been bidding for both prizes. In period $2T'+1$, $1h$'s payoff from bidding for both prizes is $-2\varepsilon + (1-\eta_{2T'+1})2V_1$, from bidding for one prize is $-\varepsilon + (1-\eta_{2T'+1})V_1$. So unless $\eta_{2T'+1} = 1-\varepsilon/V_1$, in which case $1h$ is indifferent among bidding for any subset of prizes, $1h$ either prefers conceding or bidding for both prizes. In period $2T'$, if $\eta_{2T'} > \bar{\eta}^*_{2T'}$, then the $2l$'s payoff from bidding for both is $-2\varepsilon + 2V_2 > 0$, which is larger than that from bidding for one prize: $-\varepsilon + V_2$; if $\eta_{2T'} < \bar{\eta}^*_{2T'}$ and $\rho_{2T'} < \bar{\rho}^*_{2T'}$, then $2l$'s payoff from bidding for both is $-2\varepsilon + (1-\rho_{2T'})2V_2 > 0$ and again is larger than the payoff from bidding for one: $-\varepsilon + (1-\rho_{2T'})V_2$; if $\eta_{2T'} < \bar{\eta}^*_{2T'}$ and $\rho_{2T'} > \bar{\rho}^*_{2T'}$, since $2h$ always bids for both, if $2l$ bids for one prize, then she reveals her budget and in the next period $1h$ will bid for sure, which leads to a total payoff $-\varepsilon$, so $2l$ will randomize between conceding and bidding for both prizes. In period $2T'-1$, if $\eta_{2T'-1} > \bar{\eta}^*_{2T'-1}$, then $1h$ and

$1l$ concede both since the payoff from bidding for both (one) is -2ε ($-\varepsilon$). If $\eta_{2T'-1} \leq \bar{\eta}^*_{2T'-1}$ and $\rho_{2T'-1} > \bar{\rho}^*_{2T'-1}$, there are two cases:

- If $(1-\varepsilon/V_1)^2 < \eta_{2T'-1} \leq \bar{\eta}^*_{2T'-1}$, then both $1h$ and $1l$'s payoffs from bidding for one is $-\varepsilon$ since starting from the next period players compete for one prize and $2l$ has 2ε left, which implies that the threshold belief for player 2 to concede is $(1-\varepsilon/V_1)^2$. On the other hand, both $1h$ and $1l$'s payoffs from bidding for both are $-2\varepsilon + 2V_1(1-\eta_{2T'-1})\beta^{2l}_{2T'}(2)$, where $\beta^{2l}_{2T'}(k)$ is the probability that $2l$ concedes $k = 0,1,2$ prizes in period $2T'$. Since the posterior belief satisfies $\eta_{2T'+1} = 1-\varepsilon/V_1$, $1h$ and $1l$ obtain a positive payoff from bidding for both (strictly positive when $\eta_{2T'-1} < \bar{\eta}^*_{2T'-1}$).
- If $\eta_{2T'-1} < (1-\varepsilon/V_1)^2$, then both $1h$ and $1l$'s payoffs from bidding for both are still $-2\varepsilon + 2V_1(1-\eta_{2T'-1})\beta^{2l}_{2T'}(2) > 0$. If $1h$ bids for one, then $2l$ will concede with positive probability β^{2l}_{2T-2} in the next period so that the posterior jumps to $(1-\varepsilon/V_1)^2$, but the unconditional probabilities of concession satisfy $(1-\eta_{2T'-1})\beta^{2l}_{2T'}(2) > (1-\eta_{2T'-1})\beta^{2l}_{2T-2}$. As a result, we have $-2\varepsilon + 2V_1(1-\eta_{2T'-1})\beta^{2l}_{2T'}(2) > -\varepsilon + (1-\eta_{2T'-1})\beta^{2l}_{2T-2}V_1$ whenever $-2\varepsilon + 2V_1(1-\eta_{2T'-1})\beta^{2l}_{2T'}(2) \geq 0$.

Therefore, in both cases, bidding for one prize is strictly inferior for both $1h$ and $1l$. Finally, if $\eta_{2T'-1} < \bar{\eta}^*_{2T'-1}$ and $\rho_{2T'-1} < \bar{\rho}^*_{2T'-1}$, then by a similar argument as in the previous case, $1h$ bids for both and $1l$ randomizes between bidding for and conceding both, since bidding for one reveals her budget, which leads to a negative payoff. Continuing the above argument backwards, we conclude that when $n = 2$, other than in periods $2T'$ and $2T'+1$, on the equilibrium path both players will either concede or bid for both prizes.

Notice that whenever a player concedes one prize, the other player can bid in more periods in the subsequent competition for the remaining prize. Since for any given remaining length of time that a low budget player can bid, the thresholds for the two-prize problem are the same as those for the one-prize problem, this implies that whenever a player concedes one prize, the other player either strictly prefer to bid for the remaining prize in the next period, or will concede with a lower probability than that in the two-prize case. In fact, this observation generalizes to any finite number of prizes.

The remaining step is to make sure that with $n \geq 3$ available prizes, no player on the equilibrium path will concede any non-empty subset of prizes. Suppose this is true for k prizes for all $k = 1,2,\ldots,n$. When there are $n+1$ prizes, if player i ever concedes $m < n+1$ prizes before the low budget players deplete their budgets, then in the subsequent competition for the remaining

$n+1-m \leq n$ prizes, by the induction hypothesis, both players will either concede or bid for all. In particular, after player i conceding m prizes, the low budget player $j \neq i$ with a remaining budget $\hat{B}_l$ can continue to compete for $S[n+1-m] \equiv 2\hat{B}_l/((n+1-m)\varepsilon)$ periods, whereas she can only compete for $S[n] \equiv \hat{B}_l/(n\varepsilon)(< S[n+1-m])$ periods if player i were to bid for n prizes. This lowers the relevant threshold beliefs from $(1-\varepsilon/V_2)^{S[n]/2}$ to $(1-\varepsilon/V_2)^{S[n+1-m]/2}$ for player 1, and from $(1-\varepsilon/V_1)^{S[n]/2}$ to $(1-\varepsilon/V_1)^{S[n+1-m]/2}$ for player 2. Therefore, if a low-budget player j were to randomize had player i bid for all $n+1$ prizes (which always holds in equilibrium), after player i's concession of m prizes, player j will strictly prefer to bid for all remaining prizes, in which case the low-budget player j's continuation payoff is zero. Hence, the low-budget player j obtains a negative payoff $-(n+1-m)\varepsilon$ if she concedes m prizes. That is, in equilibrium if $m > 0$ then $m = n+1$. □

Proof of Proposition 5.1. Let $\tau(B) \leq B$ denote the concession time for a player with budget $B \in [\underline{B}, \bar{B}]$, where $\tau(\cdot)$ is non-decreasing. For $t \geq 0$, define $G(t) \equiv \Pr(\{B : \tau(B) \leq t\})$ as the probability that a player has conceded by time t. We will first show that $G(\cdot)$ is strictly increasing, has no atoms, and satisfies $\min\{t : G(t) = 1\} = \bar{B}/c$. That is, $G(\cdot)$ admits a density $g(\cdot)$ almost everywhere and $\tau(\cdot)$ is in fact strictly increasing. The proof is divided into three steps:[41]

(1) $\tau(\bar{B}) = \bar{B}/c$. If not, then for a type B that is close enough to $\bar{B}$, she can profitably deviate to conceding slightly after $\tau(\bar{B})$.

(2) There are no jumps in $G(\cdot)$. If there were a jump at some time t, then there will be types who are suppose to concede at t, but they have budgets to wait slightly longer in order to win with a discrete probability, which is a profitable deviation.

(3) There is no interval (t', t'') with $t'' < \bar{B}/c$ such that $G(\cdot)$ is constant on this interval. If there were such an interval, then those types who concede at (or right after) t'' will prefer to concede at (or right after) t'.

We next show that under the monotone hazard rate condition, there is a unique cutoff budget $B^* \in (\underline{B}, \bar{B})$ such that for any $B < B^*$, a player with budget B is indifferent between conceding at any time before she depletes her budget; a player with budget $B > B^*$ only concedes after she runs out of budget. Since players' values are complete information, for a player to concede before depleting her budget, i.e., $\tau(B) < B/c$, she must be indifferent between bidding and conceding at

[41] This argument is related to the proof in Abreu and Gul (2000) that shows uniqueness of the concession probability distribution in a reputational bargaining with two-sided behavioral types.

any time between 0 and B/c. The indifference condition is

$$V \cdot \lambda(B) \cdot \frac{1}{\tau'(B)} \cdot dt = c \cdot dt,$$

where the left-hand side of the equation is the expected benefit from waiting for a length dt of time and the right-hand side is the cost of waiting. Therefore, players' concession rates are constant if they concede before running out of budget. Since $\lambda(B)$ is increasing in B, so is $\tau'(B)$. That is, $\tau(B)$ is increasing and convex for those B's such that $\tau(B) < B/c$. Since $\lambda(B)/\tau'(B) = g(\tau(B))/(1 - G(\tau(B))) \equiv \lambda_G(\tau(B))$ and $\lim_{t \to \bar{B}/c} \lambda_G(t) = \infty$, there exists t^* such that for any $t > t^*$, $V\lambda_G(t) > c$. Therefore, for a player with a budget close to $\bar{B}$, at the time she is depleting her budget she strictly prefers waiting instead of conceding; as a result, she only concedes when she runs out of budget. Moreover, suppose a player with budget B concedes at time B/c and $G(\cdot)$ is differentiable at B/c, then $\lambda(B)/\tau'(B) = \lambda(B)c$. Since $\lambda(B)$ is strictly increasing, we have $V\lambda(B')c > c$ for all $B' > B$, it follows that a payer with budget B' will concede at time B'/c. This establishes that there is a cutoff B^* such that $\tau(B) = B/c$ for all $B \geq B^*$. The uniqueness of B^* follows directly from the monotone hazard rate assumption.

Finally, since a player with budget $\underline{B}$ will concede immediately in a monotone equilibrium, by solving the differential equation from the above indifference condition, we have identified a unique monotone equilibrium as described in the proposition. □

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